Half Angles

$\because$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\therefore$

$$\sin 2\,\theta = 2\,\sin\theta\,\cos\theta \qquad\qquad \sin\theta = 2\,\sin\frac{\theta}{2}\,\cos\frac{\theta}{2}$$

$$\cos 2\,\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \qquad \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$$

$$\tan 2\,\theta = \frac{2\,\tan\theta}{1-\tan^2\theta} \qquad\qquad \tan\theta = \frac{2\,\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$\because \quad \cos\theta = 1 - 2\sin^2\dfrac{\theta}{2} \quad \therefore \sin^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{2}$ $\qquad$ $\boxed{\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}}$

$\because \quad \cos\theta = 2\cos^2\dfrac{\theta}{2} - 1 \quad \therefore \cos^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{2}$ $\qquad$ $\boxed{\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}}$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \cdot \frac{2\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \frac{\sin\theta}{2\cos^2\frac{\theta}{2}} = \frac{\sin\theta}{1+\cos\theta} \qquad \boxed{\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}} \quad \ldots (1)$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \cdot \frac{2\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{2\sin^2\frac{\theta}{2}}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta} \qquad \boxed{\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}} \quad \ldots (2)$$

$$\tan\frac{\theta}{2} = \frac{\tan\theta}{\tan\theta} \cdot \frac{\sin\theta}{1+\cos\theta} = \frac{\tan\theta\sin\theta}{\tan\theta+\sin\theta} \qquad \boxed{\tan\frac{\theta}{2} = \frac{\tan\theta\sin\theta}{\tan\theta+\sin\theta}}$$

$$\tan\frac{\theta}{2} = \frac{\tan\theta}{\tan\theta} \cdot \frac{1-\cos\theta}{\sin\theta} = \frac{\tan\theta-\sin\theta}{\tan\theta\sin\theta} \qquad \boxed{\tan\frac{\theta}{2} = \frac{\tan\theta-\sin\theta}{\tan\theta\sin\theta}}$$

Let $\quad t = \tan\dfrac{\theta}{2}\,,$ $\qquad$ so $\quad \boxed{\tan\theta = \dfrac{2t}{1-t^2}}$

$$(1)\times(2): \quad t^2 = \frac{\sin\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$t^2 + t^2\cos\theta = 1 - \cos\theta\,, \quad \cos\theta + t^2\cos\theta = 1 - t^2 \qquad \boxed{\cos\theta = \frac{1-t^2}{1+t^2}}$$

$$\sin\theta = \tan\theta\cos\theta = \frac{2t}{1-t^2} \cdot \frac{1-t^2}{1+t^2} = \frac{2t}{1+t^2} \qquad \boxed{\sin\theta = \frac{2t}{1+t^2}}$$