Sequences: Ordered numbers following a rules. The terms are denoted by $T_1, T_2, \ldots, T_n, \ldots$

(Common notations: $a = T_1$ and $l = T_n$)

Series: Sum of terms in a Sequence: $S_n = \sum_{k=1}^{n} T_k = T_1 + T_2 + \ldots + T_{n-1} + T_n$

Arithmetic Progressions (APs):

$T_n - T_{n-1} = d$, which is called the *Common Difference.*

A simple test: The sequence is an AP if $T_3 - T_2 = T_2 - T_1 = d$, or simply $T_2 = \dfrac{T_3 + T_1}{2}$.

$$\begin{aligned}
\because\ & T_n - T_{n-k} = kd \\
& T_n - T_1 = T_n - T_{n-(n-1)} = (n-1)\,d \\
& T_n - a = (n-1)\,d \\
\therefore\ & \boxed{T_n = a + (n-1)\,d} \quad \text{where } T_n \text{ is the ``n''th term.} \\
\text{or}\ & l = a + (n-1)\,d
\end{aligned}$$

$T_{k+2} - T_{k+1} = T_{k+1} - T_k\ (= d)$ for any positive integer $k \leq n-2 \quad \Leftrightarrow \quad T$ is in AP.

$\dfrac{T_k + T_{k+2}}{2} = T_{k+1}$, which is called the arithmetic mean of T_k and T_{k+2}.

i.e. $\boxed{\text{The arithmetic mean of } a \text{ and } b \text{ is } \dfrac{a+b}{2}.}$

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} T_k = T_1 + T_2 + \ldots + T_{n-1} + T_n \\
&= a + (a+d) + (a+2d) + \ldots + [a + (n-2)d] + [a + (n-1)d] \\
&= na + d\,[1 + 2 + \ldots + (n-2) + (n-1)] \\
&= na + d\,\frac{n(n-1)}{2} \\
&= \tfrac{1}{2}n\big[2a + (n-1)d\big] \\
\therefore\ & \boxed{S_n = \tfrac{1}{2}n\big[2a + (n-1)d\big]} \quad \text{where } S_n \text{ is the sum of the first ``n''th terms of the sequence.}
\end{aligned}$$

$$\begin{aligned}
l &= a + (n-1)\,d \\
a + l &= 2a + (n-1)\,d \\
\therefore\ & \boxed{S_n = \tfrac{1}{2}\,n\,(a+l)}
\end{aligned}$$

Geometric Progressions (GPs):

$$\frac{T_n}{T_{n-1}} = r \,, \quad \text{which is called the } \textit{Common Ratio.}$$

$$\because \quad \frac{T_n}{T_{n-k}} = r^k$$

$$\frac{T_n}{T_1} = \frac{T_n}{T_{n-(n-1)}} = r^{n-1}$$

$$\frac{T_n}{a} = r^{n-1}$$

$$\therefore \quad \boxed{T_n = ar^{n-1}}$$

$$\text{or} \quad l = ar^{n-1}$$

$$\frac{T_{k+2}}{T_{k+1}} = \frac{T_{k+1}}{T_k} \; (= r) \text{ for any positive integer } k \leq n-2 \quad \Leftrightarrow \quad T \text{ is in } GP.$$

$$\sqrt{T_k \cdot T_{k+2}} = \pm T_{k+1}, \text{ which is called the geometric mean of } T_k \text{ and } T_{k+2}.$$

$$\text{i.e.} \quad \boxed{\text{The geometric mean of } a \text{ and } b \text{ is } \sqrt{ab} \text{ or } -\sqrt{ab}\,.}$$

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} T_k = T_1 + T_2 + \ldots + T_{n-1} + T_n \\
&= a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \\
&= a\,(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1}) \\
&= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

$$\therefore \boxed{S_n = \frac{a(1-r^n)}{1-r}} \quad \left(\text{or } \frac{a(r^n-1)}{r-1} \text{ when } r > 1 \ \ldots \text{ if you really hate negatives.}\right)$$

$$\text{and} \quad \boxed{S_\infty = \frac{a}{1-r} \quad \text{where } |r| < 1}$$

$$\begin{aligned}
l &= ar^{n-1} \\
a - lr &= a - ar^n = a\;1 - r^n)
\end{aligned}$$

$$\therefore \boxed{S_n = \frac{a - rl}{1-r}}$$

If T_k is in AP with common difference d, then a^{T_k} is in GP with common ratio a^d.

$$\left(a^{T_{k+1}} = a^{T_k + d} = a^{T_k} \cdot a^d\right)$$

If T_k is in GP with common ratio r, then $\log_a T_k$ is in AP with common difference $\log_a r$.

$$(\log_a T_{k+1} = \log_a (T_k \cdot r) = \log_a T_k + \log_a r)$$