Rates of Change

Interdependent variables on time:

If $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$

Likewise, if $\dfrac{dy}{dx}$ is known, $g'(t) = \dfrac{dy}{dx} \cdot f'(t)$, and $f'(t) = \dfrac{g'(t)}{\frac{dy}{dx}}$, or in chain-rule form

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} , \quad \text{and} \quad \frac{dx}{dt} = \frac{dy}{dt} \Big/ \frac{dy}{dx}$$

Initial Conditions: To determine a measurement at time t, you need the rate of change and a boundary condition, which is usually the *initial condition* – the measurement when $t = t_0$.

Given $\displaystyle\int f'(t)\, dt = f(t) + C$

$$\int_{t_0}^{t_1} f'(t)\, dt = \Big[f(t)\Big]_{t_0}^{t_1} = f(t_1) - f(t_0) , \quad C = -f(t_0) \text{ is the initial condition.}$$

$$\boxed{f(T) = f(t_0) + \int_{t_0}^{T} f'(t)\, dt} \quad \text{gives the measurement at time } T$$

In practice, for measurement $U(t)$ with rate of change $u(t)$, first integrate $u(t)$ and get the integral $I(t)$, so $U(t) = I(t) + C$, then determine C by substituting the boundary condition $U(t_0)$.
$C = U(t_0) - I(t_0)$, then finally derive the measurement:
$U(t) = I(t) - I(t_0) + U(t_0)$

The boundary condition may be at $+\infty$ (e.g. when a portion of the measurement is a constant over time). In this case,

$$\int_{t_1}^{+\infty} f'(t)\, dt = \Big[f(t)\Big]_{t_1}^{+\infty} = f(+\infty) - f(t_1)$$

$$f(T) = f(+\infty) - \int_{T}^{+\infty} f'(t)\, dt \qquad \text{gives the measurement at time } T$$

Exponential growth and decay: $\boxed{\dfrac{d}{dt}f(t) = kf(t) \quad \Leftrightarrow \quad f(t) = f(0)\, e^{kt}\ , \quad \text{where } k, f(t) \in \mathbb{R}}$

Note: When $t = 0\ , \quad f(0) = f(0)\, e^0\ .$

$f(t)$ is in exponential growth when $k > 0$ (as $\lim\limits_{t \to +\infty} f(t) = +\infty$ when $c > 0$, or $-\infty$ when $c < 0$)

or exponential decay when $k < 0$ (as $\lim\limits_{t \to +\infty} f(t) = 0$)

(When $k = 0, f(t) = f(0)$. i.e. the function is a constant – independent of time.)

This property means the rate of change is proportional to the function value, so

$\boxed{y = y_0 e^{kt}}$ where $y = f(t)$ and $y_0 = f(0)$

If k is restricted to $+\mathbb{R}$, i.e. $k > 0$, then exponential decay can be presented as $y = y_0 e^{-kt}$

Natural Growth and GPs: For $t = 0, 1, 2, \ldots$ $y_0, y_1, y_2, \ldots$ become $V_0, V_0 e^k, V_0 e^{2k}, \ldots$

So natural growth is a GP with $a = V_0$ and $r = e^k$.

Also, the difference between any two consecutive terms (the gain or loss) is

$V_0 e^{(n+1)k} - V_0 e^{nk} = V_0\, (e^k - 1)\, e^{nk}$,

which is also a GP, with $a = V_0\, (e^k - 1)$ and $r = e^k$.

The above partially explains why the derivative of an exponential function is proportional to itself.