Interest

Sequences and Series:

AP: $T_n = a + (n-1)\,d\,, \quad S_n = \frac{1}{2}n\big[2a + (n-1)d\big] = \frac{1}{2}\,n\,(a+l)$

GP: $T_n = ar^{n-1}\,, \quad S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a-lr}{1-r}\,, \quad S_\infty = \dfrac{a}{1-r}$ where $|r|<1$

Common Notations: (All are positive.)

P: Principle

R: Interest Rate

I: Interest

N or n: Number of Periods

M: Payment per Period

A_n: Balance at the *end* of the Period n

Simple Interest: APs with $\quad a = P + PR\,, \quad d = PR$

$I = PR\,, \quad A_{k+1} = A_k + I = A_k + PR$

$A_0 = P \quad$ (A_0 is not in the sequence, which starts from A_1.)

$A_1 = A_0 + PR = P + PR = P(1+R)$

$A_2 = A_1 + PR = (P+PR) + PR = P + PR\cdot 2 = P(1 + R\cdot 2)$

$A_3 = A_2 + PR = (P + PR\cdot 2) + PR = P + PR\cdot 3 = P(1 + R\cdot 3)$

$\ldots$

$$\boxed{A_n = P(1+Rn)}$$

Compound Interest: GPs with $\quad a = P(1+R)\,, \quad r = 1+R$

$I = A_kR\,, \quad A_{k+1} = A_k + I = A_k + A_kR = A_k(1+R)$

$A_0 = P \quad$ (A_0 is not in the sequence, which starts from A_1.)

$A_1 = A_0(1+R) = P(1+R)$

$A_2 = A_1(1+R) = [P(1+R)]\,(1+R) = P(1+R)^2$

$A_3 = A_2(1+R) = \big[P(1+R)^2\big]\,(1+R) = P(1+R)^3$

$\ldots$

$\boxed{A_n = P(1+R)^n} \qquad$ For depreciation: $\boxed{A_n = P(1-R)^n}$

If each period is divided into h parts, over which the interest is compounded, then

$A_n = P\left(1 - \frac{R}{h}\right)^{nh} = P\left(1 - \frac{Rn}{nh}\right)^{nh} = P\left(1 - \frac{Rn}{m}\right)^{m}\,, \quad$ where $m = nh$.

$\lim\limits_{h\to\infty} A_n = \lim\limits_{m\to\infty} P(1 + \frac{Rn}{m})^m = P\,e^{Rn}$. In this case, n becomes a continuous measurement of time T.

$A_T = P\,e^RT\,, \quad A_0 = P\,, \quad$ The Interest $\quad I = A_T - P = A_T - A_0 = \left[\dfrac{dA}{dt}\right]_0^T = \displaystyle\int_0^T PR\,e^Rt\,dt\,.$

$\therefore \quad \boxed{I = \displaystyle\int_0^T PR\,e^Rt\,dt}\,, \quad$ where I is the interest compounded instantaneously;

P is the principle; R is interest rate per period; and T is the number of periods.

Regular Payments (with compound interest)

Starting with a zero balance, if a payment of M is made at the *beginning* of each period, what is the balance at the *end* of the Nth period?

Method I: Let T_k be what the payment made at the *beginning* of the "k"th period is worth at the *end* of the Nth period (when the investment matures).

$$T_k = M(1+R)^{n-k+1}$$

$$\begin{aligned} A_n &= \sum_{k=1}^{n} T_k = T_1 + T_2 + \ldots + T_{n-1} + T_n \\ &= M(1+R)^n + M(1+R)^{n-1} + \ldots + M(1+R)^2 + M(1+R) \\ &= M(1+R) \cdot \left[(1+R)^{n-1} + (1+R)^{n-2} + \ldots + (1+R) + 1\right] \\ &= M(1+R) \cdot \frac{(1+R)^n - 1}{(1+R) - 1} \end{aligned}$$

$$\therefore \boxed{A_n = M \cdot \frac{(1+R) \times [(1+R)^n - 1]}{R}}$$ ... how much your investment is worth if you invest M per period for n periods.

$$\boxed{M = A_n \cdot \frac{R}{(1+R) \times [(1+R)^n - 1]}}$$ ... how much to invest per period if you want to receive A_n after n periods

Method II: Let A_k be the balance at the *end* of the "k"th period (which is what the bank statement would show).

$$\begin{aligned} A_1 &= M(1+R) \\ A_2 &= (A_1 + M)(1+R) = [M(1+R) + M] \cdot (1+R) = M\left[(1+R)^2 + (1+R)\right] \\ A_3 &= (A_2 + M)(1+R) = \left(M\left[(1+R)^2 + (1+R)\right] + M\right) \cdot (1+R) = M\left[(1+R)^3 + (1+R)^2 + (1+R)\right] \\ &\ldots \\ A_n &= M \cdot \left[(1+R)^n + (1+R)^{n-1} + \ldots + (1+R)\right] \\ &= M(1+R) \cdot \left[(1+R)^{n-1} + (1+R)^{n-2} + \ldots + (1+R) + 1\right] \end{aligned}$$

which will give the same result as in Method I.

Paying Off a Loan (with compound interest)

Method I: Let's consider investing by regular payments aiming at receiving an amount at the end of n periods to pay off a loan of P. The investment calculation differs from the "Regular Payments" formula as the payment is made at the *end* of the period, i.e. no payment at the beginning of the first period, but an extra payment is made at the end of the last. Still n payments of M have been made but the amount is $[M(1+R)^n - M]$ less to compensate the interest for that "late" payment. The $Loan = Investment$ formula becomes:

$$P(1+R)^n = M \cdot \frac{(1+R) \times [(1+R)^n - 1]}{R} - [M(1+R)^n - M] = M \cdot \frac{(1+R)^n - 1}{R}$$

$$\therefore \boxed{M = \frac{PR(1+R)^n}{(1+R)^n - 1} = \frac{PR}{1 - \frac{1}{(1+R)^n}}}$$ ... how much to pay per period to pay off a loan of P in n periods

Method II: Let B_k be the loan balance at the *end* of the "k"th period (which is what the bank statement would show).

$$\begin{aligned} B_1 &= P(1+R) - M \\ B_2 &= B_1(1+R) - M = [P(1+R) - M] \cdot (1+R) - M = P(1+R)^2 - M(1+R) - M \\ B_3 &= B_2(1+R) - M = \left[P(1+R)^2 - M(1+R) - M\right] \cdot (1+R) - M = P(1+R)^3 - M(1+R)^2 - M(1+R) - M \\ &\ldots \\ B_n &= P(1+R)^n - M(1+R)^{n-1} - M(1+R)^{n-2} - \ldots - M(1+R) - M = 0 \quad \text{(Last period balance must be zero.)} \end{aligned}$$

$$P(1+R)^n = M \cdot \frac{(1+R)^n - 1}{(1+R) - 1} = M \cdot \frac{(1+R)^n - 1}{R}$$

which will give the same result as in Method I.