Counting

Ordered Selections *with* Repetition: If n distinct elements are to be arranged in order and repetition is allowed, there are n possibilities to select an element for the first position, and also n possibilities to select for the second position, and so on. Therefore, there are n^n possible arrangements.

$\therefore$ $\boxed{n^n = \text{Number of ordered arrangements of } n \text{ elements } \textit{with} \text{ repetition.}}$

If r elements are to be arranged in order out of n elements repetition allowed, there are r positions to fill. Therefore, there are n^r possible arrangements. (Note: $r \geq 0$; r can be larger than n .)

$\therefore$ $\boxed{n^r = \text{Number of ordered arrangements of } r \text{ elements from a set of } n \textit{ with} \text{ repetition.}}$

Permutations: A *permutation* is an ordered arrangement of elements selected from a certain set *without* repetition.

If there are n distinct elements in the set, there are n possibilities to select an element for the first position. After that, as repetition is not allowed, there are only $n-1$ remaining elements to be selected for the second position. Likewise, there are $n-2$ elements for the third position, and so on. Therefore, there are $n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$ possible permutations in total. This number is called "n factorial" — $n!$.

Definition: $\boxed{n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1 = \text{Number of permutations selected from a set of } n \text{ elements.}}$

e.g. There are $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ permutations for 5 distinct elements.

If r elements are selected out of a set of n ($r \leq n$), there are only r positions to fill. Therefore, there are $n(n-1)\ldots(n-r+1)$ permutations, which equals to $n(n-1)\ldots(n-r+1)\cdot\frac{(n-r)(n-r-1)\ldots 2\cdot 1}{(n-r)(n-r-1)\ldots 2\cdot 1} = \frac{n!}{(n-r)!}$.

Alternatively, if we line up all n elements, we can put the first r elements in one group and the last $n-r$ elements in another. There are still $n!$ permutations in total, but for each of the permutation of the first group or r, there are $(n-r)!$ permutations for the second group. These $(n-r)!$ permutations are in fact counted as one if only the first r elements is of interest. That means the "total number of permutations" ($n!$) is $(n-r)!$ times the "number of permutations of r elements out of n" (${}^nP_r$). So $(n-r)!\cdot({}^nP_r) = n!$.

Definition: $\boxed{{}^nP_r = \dfrac{n!}{(n-r)!} = \text{Number of permutations of } r \text{ elements selected from a set of } n \text{ elements } \textit{without} \text{ repetition.}}$

e.g. There are ${}^5P_3 = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = 60$ permutations of 3 elements selected from a set of 5 elements.

Note: $\because \quad n! = {}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$, $\quad\therefore\quad 0! = \frac{n!}{n!}$. $\boxed{0! = 1}$

Combinations: A *combination* is an unordered subset of elements selected from a certain set.

With reference to the analysis of permutations (ordered arrangements), since there are ${}^nP_r$ ways to arrange r elements from a set of n, if their order is *not* of interest, for each selection of r elements, the $r!$ permutations of them are in fact the same selection and therefore counted as one. So the "number of permutations" (${}^nP_r$) is $r!$ times the "number of combinations" (${}^nC_r$). So $r! \cdot ({}^nC_r) = {}^nP_r = \frac{n!}{(n-r)!}$.

Definition: $\boxed{{}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \text{Number of combinations of } r \text{ elements selected from a set of } n \text{ elements.}}$

e.g. There are ${}^5C_3 = \dfrac{5!}{3!(5-3)!} = \dfrac{5!}{3!2!} = 10$ combinations of 3 elements selected from a set of 5 elements.

Note : $\because \quad {}^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{[n-(n-r)]!(n-r)!} = {}^nC_{n-r}\ , \quad \therefore \quad \boxed{{}^nC_r = {}^nC_{n-r}}$.

i.e. There are as many ways to "include" r elements from a set of n as to "exclude" $n-r$ elements from it.

Groups: When some elements need to be together in the arrangement, each group is taken as one element, and the total number of permutations is $\boxed{n! \prod_{r=1}^{n} g_r!}$, where n is the number of groups (individual elements are taken as groups of one), and g_r is the number of elements in group r (which might be 1).

e.g. Ways to arrange letters *abcdefg* with *ab* and *def* together: We have four groups — "*ab*", "*c*", "*def*" and "*g*". So $g_1 = 2,\ g_2 = 1,\ g_3 = 3,\ g_4 = 1$ and $n = 4$. There are $4! \times (2! \times 1! \times 3! \times 1!) = 288$ ways.

Separation: When there are r elements that need to be separated in the arrangement of n ($r \le n+1$), first place the $n-r$ elements that do not need to be separated, then place the remaining r elements into the $n-r+1$ "slots" (between each neighbouring elements already placed, and the two ends). The total number of permutations is $\boxed{(n-r)!\ {}^{(n-r+1)}P_r}$.

e.g. Ways to arrange letters *abcdefg* with vowels ("*ae*") separated: $n = 7,\ r = 2$. Place the 5 elements "*bcdfg*" first $\big((n-r)! = 5! \text{ ways}\big)$, then place a and e into the 6 "slots". There are $5! \times {}^6P_2 = 3600$ ways.