Continuous Probability Distributions

Continuous Random Variables: A random variable X is **continuous** iff $F_X(x)$ is continuous,

where the Cumulative Distribution Function $\boxed{F_X(x) = P(X \leq x) \text{ for } x \in \mathbb{R}.}$ For an interval $[a, b]$,

$\boxed{P(a < X \leq b) = F_X(b) - F_X(a)}$ where $P(a < X \leq b)$ is the probability of $X \leq b$ but not $X \leq a$.

Probability Density Function: $\boxed{f_X(x) = \frac{d}{dx} F_X(x), \quad x \in \mathbb{R}}$, if $F_X(x)$ is differentiable at $x = a$.

Otherwise, $f_X(x) = \lim_{x \to a^-} \frac{d}{dx} F_X(x)$, if it exists. F_X is non-decreasing and $\lim_{x \to \infty} F_X(x) = 1$.

(If there is only one random variable in the context, the subscript X in F_X and f_X can be omitted.)

$f(x) \geq 0$ for all x and $\boxed{\int_{-\infty}^{\infty} f(x)dx = 1.}$

Theorem: $F(x) = \int_{-\infty}^{x} f(t)dt.$

$P(a \leq X \leq b) = P(a < X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx.$

$\boxed{\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)dx.}$

$\boxed{\sigma^2 = \mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx.}$

$\boxed{\sigma = \mathrm{SD}(X).}$

$Y = g(X), \quad \mathrm{E}(Y) = \mathrm{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$

Theorem: $\mathrm{Var}(X) = \mathrm{E}(X^2) - \big(\mathrm{E}(X)\big)^2.$

Proof: Let $\mu = \mathrm{E}(X)$, then $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x)dx$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - 2\mu \int_{-\infty}^{\infty} x f(x)dx + \mu^2 \int_{-\infty}^{\infty} f(x)dx = \mathrm{E}(X^2) - 2\mu \cdot \mu + \mu^2 = \mathrm{E}(X^2) - \big(\mathrm{E}(X)\big)^2.$$

Lemma : $\mathrm{E}\left(\sum_{r=0}^{n} a_r X^r\right) = \sum_{r=0}^{n} a_r \mathrm{E}(X^r)$

Proof: $\text{LHS} = \int_{-\infty}^{\infty} \left(\sum_{r=0}^{n} a_r x^r\right) f(x)dx = \sum_{r=0}^{n} a_r \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{r=0}^{n} a_r \mathrm{E}(X^r) = \text{RHS}.$

Theorem: $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$, $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$, $\mathrm{SD}(aX + b) = |a| \mathrm{SD}(X)$. ($a$ and b are constants.)

(Proof as in Discrete Probability Distributions.)

Theorem: if $Z = \frac{X - \mu}{\sigma}$ where $\mu = \mathrm{E}(X)$ and $\sigma = \sqrt{\mathrm{Var}(X)}$, then $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

Proof: $\mathrm{E}(Z) = \mathrm{E}\left(\frac{X - \mu}{\sigma}\right) = \frac{\mathrm{E}(X) - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0.$

$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2} \mathrm{Var}(X) = 1.$

The Normal Distribution $\mathrm{N}(\mu, \sigma^2)$: $\boxed{\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \text{where } -\infty < x < \infty.}$

$$\int_{-\infty}^{\infty} \phi(x)dx = 1.$$

Theorem: $\mathrm{E}(X) = \mu$.

Proof: Let $z = \frac{x-\mu}{\sigma}$. $\mathrm{E}\left(\frac{X-\mu}{\sigma}\right) = \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \cdot \phi(x)dx = \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$$= \int_{x=-\infty}^{x=\infty} z \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}z^2} \sigma dz = \frac{\sigma}{\sqrt{2\pi\sigma^2}} \left(\int_{-\infty}^{0} z \cdot e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z \cdot e^{-\frac{1}{2}z^2} dz \right)$$

$$= \frac{\sigma}{\sqrt{2\pi\sigma^2}} \left(\int_{\infty}^{0} -(-z) \cdot e^{-\frac{1}{2}(-z)^2} dz + \int_{0}^{\infty} z \cdot e^{-\frac{1}{2}z^2} dz \right) = \frac{\sigma}{\sqrt{2\pi\sigma^2}} \left(\int_{0}^{\infty} (-z) \cdot e^{-\frac{1}{2}(-z)^2} dz + \int_{0}^{\infty} z \cdot e^{-\frac{1}{2}z^2} dz \right) = 0.$$

So $\mathrm{E}\left(\frac{X-\mu}{\sigma}\right) = \frac{\mathrm{E}(X)-\mu}{\sigma} = 0.$ $\quad \therefore \mathrm{E}(X) = \mu.$

$$P(X \le x) = F_X(x) = \int_{-\infty}^{x} \phi(t)dt.$$

For $Z = \frac{X-\mu}{\sigma}$, $\mu_Z = \mathrm{E}(Z) = 0$ and $\sigma_Z = \mathrm{Var}(Z) = 1$, $P(Z \le z) = F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$

The Exponential Distribution $\mathrm{Exp}(X)$: $\boxed{f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0. \end{cases}}$

$\boxed{\mathrm{E}(X) = \frac{1}{\lambda}.}$

Proof: $\mathrm{E}(X) = \int_{-\infty}^{\infty} tf(t)dt = 0 + \int_{0}^{\infty} t\lambda e^{-\lambda t} dt = \int_{0}^{\infty} (-t) \cdot (-\lambda) e^{-\lambda t} dt$

$$= \Big[(-t) \cdot e^{-\lambda t}\Big]_0^\infty - \int_0^\infty e^{-\lambda t}(-1)dt = \Big[(-t) \cdot e^{-\lambda t}\Big]_0^\infty - \frac{1}{\lambda}\Big[e^{-\lambda t}\Big]_0^\infty$$

$$= 0 - \frac{1}{\lambda}\Big[0 - 1\Big] = \frac{1}{\lambda}.$$

$\boxed{\sigma^2 = \mathrm{Var}(X) = \frac{1}{\lambda^2}.}$

Proof: $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 = \int_{-\infty}^{\infty} t^2 f(t)dt - \mu^2 = 0 + \int_0^\infty (-t^2) \cdot (-\lambda) e^{-\lambda t} dt - \mu^2$

$$= \Big[(-t^2) \cdot e^{-\lambda t}\Big]_0^\infty - \int_0^\infty e^{-\lambda t}(-2t)dt - \mu^2 = 0 + \frac{2}{\lambda}\left(\int_0^\infty t\lambda e^{-\lambda t} dt\right) - \mu^2$$

$$= \frac{2}{\lambda} \cdot \mu - \mu^2 = \frac{2}{\lambda} \cdot \frac{1}{\lambda} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$