Polynomials

General Form: $P(x) \equiv \sum_{r=0}^{n} a_k x^k = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \,,$ where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

Notations: $\deg P(x) \equiv n$

$q|r$ where $q \neq 0$ if and only if $r = kq$ for some $k \in \mathbb{Z}$. (It follows that $q|0$ for any integer $q \neq 0$.)

$\gcd(p,q) \equiv g$ where $g \in \mathbb{Z}$ is the largest number that satisfy $g|p$ and $g|q$ (the HCF of p and q).

$\gcd(p,q) = 1$ means that p and q are relative prime (and fraction $\frac{p}{q}$ cannot be simplified further).

Division Form: $P(x) \equiv A(x)Q(x) + R(x) \,,$ where $A(x)$ is the devisor, $Q(x)$ is the quotient, and $R(x)$ is the remainder.

$\deg P(x) = \deg A(x) + \deg Q(x) \,, \quad \deg P(x) \geq \deg Q(x) \,, \quad \deg R(x) < \deg Q(x)$

Remainder Theorem: If $P(x)$ is divided by $(x-\alpha)$, then the remainder is $P(\alpha)$.

Proof: $\because \quad P(x) = A(x)(x-\alpha) + R \,, \quad (\deg R(x) = 0) \,, \qquad \therefore \quad P(\alpha) = A(\alpha)(\alpha - \alpha) + R = R \,.$

Corollary: If $P(x)$ is divided by $(qx - p)$, then the remainder is $P\left(\frac{p}{q}\right)$.

Proof: $\because \quad P(x) = A(x)(qx-p) + R \,, \qquad \therefore \quad P\left(\frac{p}{q}\right) = A\left(\frac{p}{q}\right)\left[q\left(\frac{p}{q}\right) - p\right] + R = R \,.$

Factor Theorem: $(x-\alpha)$ is a factor of $P(x)$ *if and only if* $P(\alpha) = 0$.

The Fundamental Theorem of Algebra: Every polynomial of complex coefficient has at least one zero in the complex field.

Theorem 1: If $\frac{p}{q}$ is a zero of $P(x)$, where $p, q \in \mathbb{Z}$, $q \neq 0$, $\gcd(p,q) = 1$, and $P(x)$ has integer coefficients,

then $p|a_0$, $q|a_n$, and $qx - p$ is a factor of $P(x)$.

i.e. If a polynomial has a zero in simple fractional form, the coefficient of the lowest degree term of the polynomial is divisible by the numerator, and the highest by the denominator.

e.g. $6x^2 - 7x - 3 = (2x-3)(3x+1)$, has two zeros: $\frac{3}{2}$ and $\frac{-1}{3}$.

For the numerators, we have $3|-3$, $-1|-3$ and for the denominators $2|6$ and $3|6$.

Corollary: If $a_n = 1$, $q = \pm 1$ and p is an integer and $p|a_0$.

i.e. If a monic polynomial with integer coefficients has a rational zero, it is an integer and a divisor of a_0.

Theorem 2: If $p + \sqrt{q}$ is a zero of $P(x)$, where $p, q \in \mathbb{Z}$, $q \neq 0$, and $P(x)$ has rational coefficients,

then $p - \sqrt{q}$ is also a zero. (Likewise, if $p - \sqrt{q}$ is a zero, so is $p + \sqrt{q}$.)

i.e. If a polynomial with rational coefficients has irrational zeros, they are in conjugate pairs.

Theorem 3: If $a + ib$ is a zero of $P(x)$, where $a, b \in \mathbb{R}$, $b \neq 0$, and $P(x)$ has real coefficients, then $a - ib$ is also a zero.

i.e. If a polynomial with real coefficients has imaginery zeros, they are in conjugate pairs.

Theorem 4: Every polynomial has exactly $\deg P(x)$ linear factors over the complex field.

Proof: From the Fundamental Theorem of Algebra, $P(x)$ has at least one zero in the complex field, and therefore at least one linear factor. If it is $(x-\alpha)$, so that $P(x) = A(x)(x-\alpha)$, $A(x)$ of degree $n-1$ $\big(n = \deg P(x)\big)$ has at least one linear factor as well. By repeating the process, all n linear factors will be found.

i.e. Every polynomial equation of n degrees has exactly n roots in the complex field, including multiplicities.

Theorem 5: If $P(x) = 0$ has a root of multiplicity m, then $P'(x)$ has the same root of multiplicity of $(m-1)$.

Proof: Consider $P(x) = (x-\alpha)^m Q(x) \,, \quad P'(x) = (x-\alpha)^{m-1}\big(mQ(x) + (x-\alpha)Q'(x)\big) \,.$