# Growth and Decay

Notes:
1. While $Q$ or $Q(t)$ is used in general description or formula, other mnemonics may be used, such as N)umber, P)opulation, M)ass, T)emperature, V)olume etc.
2. Even $Q(t)$ is discrete in real life, it is taken as continuous real number in calculation, which is a good approximation when $Q(t)$ is large.
3. The parameter is always $t$ (time). Differentiation and Integration are with reference to $t$.

**The Basics:** Rate of change $\dfrac{dQ}{dt} \propto Q \quad \Rightarrow \quad \dfrac{dQ}{dt} = kQ\,.$ $\big($The measurement of $k$ is $s^{-1}$ (per second)$\big)$

When $k > 0$, it is an exponential growth; when $k < 0$, it is an exponential decay. (No change when $k = 0$.)

(In some text, $k > 0$ is assumed, so a decay is indicated as $\dfrac{dQ}{dt} = -kQ$ .)

**The Function:** $\boxed{\text{If } \dfrac{dQ}{dt} = kQ \quad \text{then} \quad Q(t) = Q_0\, e^{kt}\,, \quad \text{where } Q_0 = Q(0)}$ or $\quad Q(t) = A\, e^{kt}\,, \quad$ where $A = Q(0)$

$$\frac{dQ}{dt} = kQ$$

$$\frac{1}{Q}\, dQ = k\, dt$$

$$\int_{Q_0}^{Q} \frac{1}{Q}\, dQ = \int_0^t k\, dt$$

$$\Big[\ln Q\Big]_{Q_0}^{Q} = kt \qquad \text{(Yes, } kt \text{ is “unitless” — } s \cdot s^{-1}\text{)}$$

$$\ln \tfrac{Q}{Q_0} = kt$$

$$\tfrac{Q}{Q_0} = e^{kt}$$

$$Q = Q_0\, e^{kt}$$

$$\boxed{\text{If } Q(t) = Q_0\, e^{kt} \quad \text{then} \quad \frac{dQ}{dt} = kQ}$$

$$\frac{dQ}{dt} = \frac{d}{dt}\left(Q_0\, e^{kt}\right) = k\, Q_0\, e^{kt} = kQ$$

**Half Life of Decay:** $\boxed{t_{\frac{1}{2}} = \dfrac{1}{k} \ln \dfrac{1}{2} = -\dfrac{\ln 2}{k}}$ (Note: $k < 0$ in order to make $t_{\frac{1}{2}} > 0$ .)

When $Q = \dfrac{Q_0}{2}\,, \quad Q_0\, e^{kt} = \dfrac{Q_0}{2}\,,$

$$e^{kt} = \frac{1}{2}$$

$$kt = \ln \frac{1}{2}$$

$$t = \frac{1}{k} \ln \frac{1}{2}$$

Generally, when $Q = r\, Q_0, \quad t_r = \dfrac{\ln r}{k}$ . e.g. Population triple every 20 seconds: $20 = \dfrac{\ln 3}{k}$ or $k = \dfrac{\ln 3}{20}$ .

Another example: in microorganism splits, $r = 2$ and the time between consecutive splits is $t_2 = \dfrac{\ln 2}{k}$ .

Further ... The case of growth rate being directly proportional to the *entire* population is a special one. In real life, there may be a saturation point, where the growth or decay approaches zero. Let the population be $N$ and the *saturation* point be $P$. When $N \to P$, the rate of change $\frac{dN}{dt} \to 0$.

$$\frac{dN}{dt} \propto N - P \quad \Rightarrow \quad \frac{dN}{dt} = k(N-P)$$

Let $M = N - P$, then $\dfrac{dM}{dt} = \dfrac{dN}{dt} = k(N-P) = kM$

So $M = M_0\, e^{kt}$ , $\quad N - P = (N_0 - P)\, e^{kt}$ , $\quad \boxed{N = P + A\, e^{kt}\ , \quad \text{where } A = N_0 - P}$

Physically, the formula means that there is a part of the population being idle (as represented by $P$), and the growth or decay is based on the rest of the population only ($A = N_0 - P$).

Convergent: When $k < 0$, $\quad e^{kt}$ is getting smaller and approaches zero when $t \to +\infty$, so $N \to P$.
If $A < 0$, $N_0 < P$ and the population is initially less than $P$ then grows towards $P$.
If $A > 0$, $N_0 > P$ and the population is initially more than $P$ then decays towards $P$.

Divergent: When $k > 0$, $\quad e^{kt}$ is getting larger and approaches infinity when $t \to +\infty$.
If $A < 0$, $N_0 < P$ and the population is initially less than $P$ then decays away from $P$ towards $-\infty$.
If $A > 0$, $N_0 > P$ and the population is initially more than $P$ then grows away from $P$ towards $+\infty$.

Derived Use: When a problem is in the form of $\dfrac{dN}{dt}$ proportional to $N \pm C$ (a constant), the same can apply.
e.g. For a falling object in air, $\dfrac{dv}{dt} = g - kv$ , $\quad \dfrac{dv}{dt} = -k\left(v - \dfrac{g}{k}\right)$ , $\quad$ one can derived that
$v = \dfrac{g}{k} + \left(v_0 - \dfrac{g}{k}\right) e^{-kt}$ . If the object was initially at rest, $v_0 = 0$ and $v = \dfrac{g}{k}\left(1 - e^{-kt}\right)$ .

Newton's Law of Cooling A system hotter than its surrounding will lose heat and gradually cool down towards an equilibrium temperature, which is assumed to be the surrounding temperature $T_s$ . So it is a case of convergence ($k < 0$) with initial temperature higher than the surrounding ($T_0 > T_s$ or $A > 0$).

$$\frac{dT}{dt} = k(T - T_s)\ , \quad \boxed{T = T_s + A\, e^{kt}\ , \quad \text{where } k < 0 \text{ and } A = T_0 - T_s > 0.}$$

If the surrounding matter has a limited heat capacity (e.g. in a heat isolated tank), its temperature will follow the same transformation, with the same $T_s$ and $k$, but a different $T_0$ and therefore different $A$ (where $A < 0$ as the surrounding is heating up towards $T_s$).
(The above discussion can apply to a cooler system in a hotter room, with the sign of $A$ negated.)

Brine: Given a volume $V$ of solution containing quantity $Q$ of a certain substance, every unit of time $v$ unit of solution with $q$ of substance was let in, fully mixed, then let out, find $Q(t)$.

Every unit of time, the system is gaining $q$ but losing $Q \cdot \dfrac{v}{V}$, so $\dfrac{dQ}{dt} = q - kQ$, where $k = \dfrac{v}{V}$.

$$\frac{dQ}{dt} = -k\left(Q - \frac{q}{k}\right)\ , \quad Q = \frac{q}{k} + A\, e^{-kt}\ , \quad \text{where } A = Q_0 - \frac{q}{k} \quad \text{and} \quad k = \frac{v}{V}$$

$$Q = \frac{qV}{v} + \left(Q_0 - \frac{qV}{v}\right) e^{-\frac{vt}{V}} = V\left[\frac{q}{v} + \left(\frac{Q_0}{V} - \frac{q}{v}\right) e^{-kt}\right] = V\left[\rho + (\rho_0 - \rho)\ e^{-kt}\right]$$

$$\boxed{Q = V\left[\rho + (\rho_0 - \rho)\ e^{-kt}\right]}$$

where $\rho_0$ is the initial concentration, $\rho$ is that of the "refreshment", and $k$ is the "refreshment" ratio. The function always converges, becoming more concentrated if $\rho_0 < \rho$, i.e. $A < 0$.

Generally, $\boxed{Q = \dfrac{q}{k} + \left(Q_0 - \dfrac{q}{k}\right)\ e^{-kt}}$

if there are $q$ "new borns" every unit of time, and $k$ is the "death rate" ($0 < k < 1$).