Vertical Circular Motion

Simple Pendulum:

A string of l metres fixed at one end O, with a mess of m kg attached to the other end P, and making oscillations in a vertical plane. The tension in the string is T newtons and the inclination of the string to the vertical is θ. (P is swinging upwards when θ increases from 0 to π .)

Tangental Force $F_T = -mg\sin\theta$,   Norminal (Radial) Force $F_N = T - mg\cos\theta$.
Tangental Acceleration $a_T = -g\sin\theta = l\ddot{\theta}$,   Norminal (Radial) Acceleration $a_N = \frac{T}{m} - g\cos\theta = l\dot{\theta}^2$

From the formula of a_T , $\ddot{\theta} = -\dfrac{g}{l}\sin\theta$

When $\theta \to 0$ (e.g. $-10^\circ \le \theta \le 10^\circ$), $\sin\theta \to \theta$, and $\boxed{\ddot{\theta} \approx -\dfrac{g}{l}}$

This is close to a simple harmonic motion ($\ddot{x} = -n^2 x$) with $n = \sqrt{\dfrac{g}{l}}$. The period is $\dfrac{2\pi}{n} = 2\pi\sqrt{\dfrac{l}{g}}$

Vertical Circular Motion:

Bike in a cage: The radius of the cage is R . The normal inward force is N .
Tangental Force $F_T = -mg\sin\theta$,   Norminal (Radial) Force $F_N = N - mg\cos\theta$.
Tangental Acceleration $a_T = -g\sin\theta = R\ddot{\theta}$,   Norminal (Radial) Acceleration $a_N = \frac{N}{m} - g\cos\theta = R\dot{\theta}^2$

As a side note, recall $\dfrac{d}{d\theta}\left[\dfrac{1}{2}\dot{\theta}^2\right] = \dot{\theta}\cdot\dfrac{d}{d\theta}\left(\dfrac{d\theta}{dt}\right) = \dfrac{d}{d\theta}\left(\dfrac{d\theta}{dt}\right)\cdot\dfrac{d\theta}{dt} = \dfrac{d}{dt}\left(\dfrac{d\theta}{dt}\right) = \ddot{\theta}$, so $\boxed{\ddot{\theta} = \dfrac{d}{d\theta}\left[\dfrac{1}{2}\dot{\theta}^2\right]}$

From the formula of a_T , $-g\sin\theta = R\ddot{\theta} = R\cdot\dfrac{d}{d\theta}\left[\dfrac{1}{2}\dot{\theta}^2\right]$, $\displaystyle\int_0^\theta -g\sin\theta\, d\theta = \int_0^\theta R\cdot\dfrac{d}{d\theta}\left[\dfrac{1}{2}\dot{\theta}^2\right] d\theta$,

$g\big[\cos\theta\big]_0^\theta = \dfrac{1}{2R}\Big[(R\dot{\theta})^2\Big]_0^\theta = \dfrac{v^2 - u^2}{2R}$, $v^2 - u^2 = 2Rg(\cos\theta - 1)$, $\boxed{v^2 = u^2 - 2Rg(1 - \cos\theta)}$ . . . (1)

From the formula of $a_N \times R$, $R\frac{N}{m} - Rg\cos\theta = R^2\dot{\theta}^2 = v^2$ . . . (2)
(1) + (2) : $v^2 + R\frac{N}{m} - Rg\cos\theta = u^2 - 2Rg + 2Rg\cos\theta + v^2$, $R\frac{N}{m} = u^2 - 2Rg + 3Rg\cos\theta$,

$\boxed{N = m\dfrac{u^2}{R} - mg(2 - 3\cos\theta)}$

N must remain positive for all θ to hold the bike on the track and avoid an accident.

$N = m\dfrac{u^2}{R} - mg(2 - 3\cos\theta) > 0$, $u^2 > Rg(2 - 3\cos\theta)$.

RHS reaches its maximum of $5Rg$ when $\theta = \pi$. If $u^2 < 5Rg$, the bike will fall before reaching the top. On the other hand, if it falls when $\theta < \frac{\pi}{2}$, it simply rolls back safely. So $u^2 < Rg(2 - 3\cos\frac{\pi}{2}) = 2Rg$.

$\boxed{\text{When } 2Rg < u^2 < 5Rg, \text{ there will be an off-the-track fall (an accident).}}$