Strings and Governors

General Configuration: A light and smooth string of length L is attached to a vertical pole. One end of the string is attached to point A, which is fixed at the top of the pole, and another end to point B, h units of length under A. A ring of mass m has been threaded into the string at position P, so AP is l_1 units of length, and PB is l_2. AP is at angle θ_1 to the vertical, and PB at θ_2 .

Point P describes a horizontal circle of radius r in Uniform Circular Motion at angular speed of ω. While A is always fixed, P and B can be loose in some configurations. When B is loose, a mass M will be attached to it. The tension on AP is T_1 and that on PB is T_2.

Strings of Equal Length: Let $l = l_1 = l_2$, and $\theta = \theta_1 = \theta_2$; P and B are fixed. $r = l\sin\theta$, $h = 2l\cos\theta$.

Horizontal: $T_1\sin\theta + T_2\sin\theta = mr\omega^2 = ml\sin\theta\cdot\omega^2$, $T_1 + T_2 = ml\omega^2 \quad \ldots (1)$

Vertical: $T_1\cos\theta - T_2\cos\theta = mg$, $T_1 - T_2 = \dfrac{mg}{\cos\theta} = \dfrac{2mgl}{h} \quad \ldots (2)$

$$\frac{(1)+(2)}{2} : \boxed{T_1 = \frac{ml}{2}\left(\omega^2 + \frac{2g}{h}\right)}, \quad \frac{(1)-(2)}{2} : \boxed{T_2 = \frac{ml}{2}\left(\omega^2 - \frac{2g}{h}\right)}, \quad \boxed{\frac{T_1}{T_2} = \frac{\omega^2 h + 2g}{\omega^2 h - 2g} = \frac{l\omega^2\cos\theta + g}{l\omega^2\cos\theta - g}}$$

When $\omega^2 \geq \dfrac{2g}{h}$, $T_2 \geq 0$ and the second string will be taut.

Tensions in terms of θ: $\because\ h = 2l\cos\theta$, $\therefore$ $\boxed{T_1 = \dfrac{m}{2}\left(l\omega^2 + \dfrac{g}{\cos\theta}\right)}$, $\boxed{T_2 = \dfrac{m}{2}\left(l\omega^2 - \dfrac{g}{\cos\theta}\right)}$

Analysis: Since P and B are fixed, l, h and θ are constants. So the faster the angular speed, the higher the value of ω^2 and therefore higher tension on the strings (T_1 and T_2). While T_1 is always positive, T_2 can be negative numerically, which means the lower string is not taut. In such case, the configuration becomes a Conical Pendulum. If the strings are replaced by sticks, when T_2 is negative, the lower stick has an inward compress force in it.

Equal Length & Collar: Let $l = l_1 = l_2$, and $\theta = \theta_1 = \theta_2$; P is fixed, but B is a loose collar of mass M.
For P, the previous formulae of a fixed B still hold.

For a loose B with mass M, its vertical balance is $T_2\cos\theta = Mg$, $T_2 = \dfrac{Mg}{\cos\theta} = \dfrac{2Mgl}{h} \quad \ldots (3)$

Compare with the T_2 formula before: $\dfrac{2Mgl}{h} = \dfrac{ml}{2}\cdot\left(\omega^2 - \dfrac{2g}{h}\right)$, $\dfrac{2Mgl}{h} = \dfrac{ml\omega^2}{2} - \dfrac{mgl}{h}$,

$\dfrac{g}{h}(2M + m) = \dfrac{m\omega^2}{2}$, $\dfrac{\omega^2 h}{2g} = \dfrac{2M + m}{m} = \dfrac{2M}{m} + 1 \quad \ldots (4)$ $\boxed{\dfrac{M}{m} = \dfrac{1}{2}\left(\dfrac{\omega^2 h}{2g} - 1\right)}$

From (2): $mg = T_1\cos\theta - T_2\cos\theta = T_1\cos\theta - Mg$, $T_1\cos\theta = (M + m)g$, $\dfrac{T_1\cos\theta}{T_2\cos\theta} = \dfrac{(M + m)g}{Mg}$

$$\therefore \boxed{\frac{T_1}{T_2} = \frac{M + m}{M}} = 1 + \frac{m}{M} = 1 + 2\cdot\frac{2g}{\omega^2 h - 2g} = \frac{\omega^2 h - 2g + 4g}{\omega^2 h - 2g} = \frac{\omega^2 h + 2g}{\omega^2 h - 2g} \quad \text{(same as before)}$$

From (3): $T_2 = \dfrac{2Mgl}{h} = Ml\omega^2\cdot\dfrac{2g}{\omega^2 h}$, $\boxed{T_2 = \dfrac{Ml\omega^2}{\frac{2M}{m} + 1}}$, $T_1 = T_2\cdot\dfrac{M + m}{M}$, $\boxed{T_1 = \dfrac{(M + m)l\omega^2}{\frac{2M}{m} + 1}}$

Loose Ring of Mass: B is fixed and P is a ring threaded loosely on the string, so $T = T_1 = T_2$.

$$r = l_1 \sin\theta_1 = l_2 \sin\theta_2 \,, \quad h = l_1 \cos\theta_1 + l_2 \cos\theta_2 \,.$$

Horizontal: $T \sin\theta_1 + T \sin\theta_2 = T(\sin\theta_1 - \sin\theta_2) = mr\omega^2 \quad \ldots (5)$

Vertical: $T \cos\theta_1 - T \cos\theta_2 = T(\cos\theta_1 - \cos\theta_2) = mg \quad \ldots (6)$

$$(6) \div (5): \frac{g}{r\omega^2} = \frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_1 + \sin\theta_2} = \frac{-2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)} = -\tan\left(\frac{\theta_1-\theta_2}{2}\right) \,, \quad \boxed{\tan\left(\frac{\theta_2-\theta_1}{2}\right) = \frac{g}{r\omega^2}}$$

$$\text{From (5): } mr\omega^2 = T\sin\theta_1 + T\sin\theta_2 = T\left(\frac{r}{l_1} + \frac{r}{l_2}\right) = Tr\left(\frac{l_2+l_1}{l_1 l_2}\right) \,, \quad \boxed{T = m\omega^2 \frac{l_1 l_2}{L}}$$