Simple Harmonic Motion (SHM)

Basic Equations: $F \propto x$ but in opposite direction. $\therefore \boxed{\ddot{x} = -n^2 x}$, where $\frac{2\pi}{n}$ is the period (as will be shown later).

$F = m\ddot{x} = -mn^2 x$

$$\ddot{x} = v \cdot \frac{dv}{dx} = -n^2 x\,, \qquad -n^2 \int_{x_0}^{x} x\,dx = \int_{v_0}^{v} v \cdot dv\,,$$

$$-\,n^2 \cdot \frac{1}{2}(x^2 - {x_0}^2) = \frac{1}{2}(v^2 - {v_0}^2)\,, \quad -x^2 + {x_0}^2 = \frac{v^2}{n^2} - \frac{{v_0}^2}{n^2}\,, \quad x^2 + \frac{v^2}{n^2} = {x_0}^2 + \frac{{v_0}^2}{n^2}$$

i.e. There exists a triangle with a constant hypotenuse $A = \sqrt{{x_0}^2 + \frac{{v_0}^2}{n^2}}$,

and x and $\frac{v}{n}$ form the right angle, such that $\quad x^2 + \frac{v^2}{n^2} = A^2 \qquad \ldots (1)$

Let ϕ be the angle formed by x and A.

$$x = A\,\cos\phi\,, \quad v = \dot{x} = \frac{dx}{dt} = -A\,\sin\phi \cdot \frac{d\phi}{dt} \qquad \ldots (2)$$

From (1): $A^2 = x^2 + \frac{v^2}{n^2} = A^2\cos^2\phi + \frac{1}{n^2}\left[-A\,\sin\phi \cdot \frac{d\phi}{dt}\right]^2$

$$\cos^2\phi + \sin^2\phi \cdot \frac{1}{n^2}\left(\frac{d\phi}{dt}\right)^2 = 1\,, \quad \frac{1}{n^2}\left(\frac{d\phi}{dt}\right)^2 = \frac{1-\cos^2\phi}{\sin^2\phi} = 1\,, \quad \frac{d\phi}{dt} = n\,, \quad \phi = nt + \alpha$$

$$\therefore \boxed{x = A\,\cos(nt+\alpha)\,, \quad \text{where } A = \sqrt{{x_0}^2 + \frac{{v_0}^2}{n^2}} \text{ and } \alpha = \cos^{-1}\frac{x_0}{A}}$$

Derived Equations: Since $\cos(nt+\alpha) \in [-1,1]$, $x \in [-A, A]$, A is called the amplitude of the oscillation.

x goes through a cycle as nt is increased by 2π. Let period be T, then $nT = 2\pi$. $\boxed{T = \frac{2\pi}{n}}$

The frequency f is the number of *periods* per second. $\boxed{f = \frac{1}{T} = \frac{n}{2\pi}}$

From (1): $x^2 + \frac{v^2}{n^2} = A^2$, $\therefore \boxed{v^2 = n^2(A^2 - x^2)}$

From (2): $v = -A\,\sin(nt+\alpha)\cdot\frac{d}{dt}(nt+\alpha) = -A\,\sin(nt+\alpha)\cdot n \quad \boxed{v = -nA\,\sin(nt+\alpha)}$

$\ddot{x} = -nA\,\cos(nt+\alpha)\cdot\frac{d}{dt}(nt+\alpha) = -nA\,\cos(nt+\alpha)\cdot n\,, \quad \boxed{\ddot{x} = -n^2\,A\,\cos(nt+\alpha) = -n^2 x}$

(SHM is analogous to Circular Motion, where the angular speed $\omega = n$.)

Equilibrium Point: $\boxed{x = B + A\cos(nt+\alpha)}$, $\boxed{v^2 = n^2\left[A^2 - (x-B)^2\right]}$, and $\boxed{\ddot{x} = -n^2(x-B)}$

Centre: $x = B$

Interval: $x \in [B-A, B+A]$