Resistance Proportional to Speed

Basic Equation: $R \propto v$ but in opposite direction. $\therefore R = \lambda v$, where $\lambda < 0$

Physically, R is determined by many factors, but let's (artificially) define $k = -\frac{\lambda}{m} > 0$,

so $R = \lambda v = -mkv$ (opposite to v).

$R = -mkv$ is downwards went the particle is going up ($R < 0$ when $v > 0$),

and upwards when it is going down.

$m\ddot{x} = -W + R = -mg - mkv$ ($W = mg > 0$ but is always a downward force; $g > 0$ here)

$$\therefore \boxed{\ddot{x} = -g - kv} \quad \text{and} \quad \boxed{R = -mkv\,, \quad \text{where } k > 0}$$

Given a vertical initially upwards projectile with initial speed u, i.e. where $t = 0$, $x = 0$, $\dot{x} = u > 0$,

v in terms of t: $\boxed{v = ue^{-kt} - \frac{g}{k}\left(1 - e^{-kt}\right) = -\frac{g}{k} + \frac{g + ku}{k} \cdot e^{-kt}}$

$$\frac{dv}{dt} = \ddot{x} = -g - kv\,, \quad \frac{dt}{dv} = \frac{1}{-g - kv}\,, \quad t = \int_u^v \frac{dv}{-g - kv} = -\frac{1}{k}\big[\ln(g + kv)\big]_u^v = -\frac{1}{k}\ln\left(\frac{g + kv}{g + ku}\right)$$

$$e^{-kt} = \frac{g + kv}{g + ku}\,, \quad g + kv = (g + ku)e^{-kt}\,, \quad v = \frac{1}{k}\left(ge^{-kt} + kue^{-kt} - g\right) = ue^{-kt} - \frac{g}{k}\left(1 - e^{-kt}\right)$$

Likewise, $\boxed{t = \frac{1}{k}\ln\left(\frac{g + ku}{g + kv}\right)}$

x in terms of v: $\boxed{x = \frac{1}{k}\left[u - v + \frac{g}{k}\ln\left(\frac{g + kv}{g + ku}\right)\right]}$

$$v \cdot \frac{dv}{dx} = -g - kv\,, \quad \frac{dx}{dv} = \frac{v}{-g - kv}\,, \quad x = \int_u^v \frac{v}{-g - kv}\,dv$$

$$x = -\frac{1}{k}\int_u^v 1 - \frac{g}{g + kv}\,dv = -\frac{1}{k}\left[(v - u) - \frac{g}{k}\left[\ln(g + kv)\right]_u^v\right] = \frac{1}{k}\left[u - v + \frac{g}{k}\ln\left(\frac{g + kv}{g + ku}\right)\right]$$

x in terms of t: $\boxed{x = -\frac{g}{k}t + \left(\frac{g + ku}{k^2}\right)\left(1 - e^{-kt}\right)}$

$$x = \int_0^t v\,dt = \int_0^t \left(-\frac{g}{k} + \frac{g + ku}{k} \cdot e^{-kt}\right)dt = -\frac{g}{k}t + \left(\frac{g + ku}{k^2}\right) \cdot \left(1 - e^{-kt}\right)$$

Max height H at T: $\boxed{T = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right) = \frac{u - kH}{g}}$

At max, $t = T$ and $\frac{dx}{dt} = v = 0$, i.e. $-\frac{g}{k} + \frac{g + ku}{k} \cdot e^{-kT} = 0$

$$(g + ku)\,e^{-kT} = g\,, \quad e^{-kT} = \frac{g}{g + ku}\,, \quad -kT = \ln\left(\frac{g}{g + ku}\right)\,, \quad T = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right)$$

When $x = H, v = 0$, so

$$H = x(T) = \frac{1}{k}\left[u - 0 + \frac{g}{k}\ln\left(\frac{g + k \cdot 0}{g + ku}\right)\right] = \frac{1}{k}\left[u + \frac{g}{\not{k}}(-\not{k}T)\right] = \frac{u - gT}{k}$$

$$T = \frac{u - kH}{g}$$

Free Fall: Based on the above formulae, but with $u = 0$ and the x-axis is downward, so the signs of the x, v and $\ddot{x}$ variables need to be negated in those formulae.

e.g. $-\ddot{x} = -g - k(-v)$, $\boxed{\ddot{x} = g - kv}$

v in terms of t: $\boxed{v = \frac{g}{k}\left(1 - e^{-kt}\right)}$ and $\boxed{t = \frac{1}{k}\ln\left(\frac{g}{g - kv}\right)}$

Terminal Velocity: $v_T = \lim_{t\to+\infty} v = \lim_{t\to+\infty} \frac{g}{k}\left(1 - e^{-kt}\right) = \frac{g}{k}$

Also, $\ddot{x} = g - kv_T = 0$, $\therefore \boxed{v_T = \frac{g}{k}}$

x in terms of v: $\boxed{x = -\frac{v}{k} - \frac{g}{k^2}\ln\left(1 - \frac{kv}{g}\right)}$

x in terms of t: $\boxed{x = \frac{g}{k}t - \frac{g}{k^2}\left(1 - e^{-kt}\right)}$