Resistance Proportional to Speed Squared

Basic Equation: $R \propto v^2$ but in opposite direction, so $R = \lambda v^2 = -mkv^2$.

$$m\ddot{x} = -W + R = -mg - mkv^2$$

$$\therefore \quad \boxed{\ddot{x} = -g - kv^2} \quad \text{and} \quad \boxed{R = -mkv^2\ , \quad \text{where } k = \tfrac{\lambda}{m} > 0}$$

Given a vertical initially upwards projectile with initial speed u, i.e. when $t = 0$, $x = 0$, $\dot{x} = u > 0$,

v in terms of t: $\boxed{v = \sqrt{\tfrac{g}{k}} \tan\left(\alpha - t\sqrt{gk}\right)\ , \quad \text{where } \alpha = \tan^{-1}\left(u\sqrt{\tfrac{k}{g}}\right)}$

$$\frac{dv}{dt} = \ddot{x} = -g - kv^2\ , \quad \frac{dt}{dv} = \frac{1}{-g-kv^2}\ , \quad t = \int_u^v \frac{dv}{-g-kv^2} = \frac{-1}{\sqrt{gk}}\left[\tan^{-1} v\sqrt{\tfrac{k}{g}}\right]_u^v$$

$$-t\sqrt{gk} = \tan^{-1} v\sqrt{\tfrac{k}{g}} - \tan^{-1} u\sqrt{\tfrac{k}{g}}\ , \quad -tk\psi = \tan^{-1}\tfrac{v}{\psi} - \tan^{-1}\tfrac{u}{\psi}\ , \text{ where } \psi = \sqrt{\tfrac{g}{k}}$$

$$\tan(-tk\psi) = \frac{\frac{v}{\psi} - \frac{u}{\psi}}{1 + \frac{v}{\psi}\cdot\frac{u}{\psi}}\ , \qquad -\psi\tan(tk\psi) = \frac{v-u}{1+\frac{vu}{\psi^2}}\ , \qquad -\psi\tan(tk\psi)(1 + \tfrac{vu}{\psi^2}) = v - u\ ,$$

$$-\psi\tan(tk\psi) - \tan(tk\psi)\frac{vu}{\psi} = v - u\ , \qquad v[1 + \tfrac{u}{\psi}\tan(tk\psi)] = u - \psi\tan(tk\psi) = \psi\left(\tfrac{u}{\psi} - \tan(tk\psi)\right)\ ,$$

$$v = \psi\cdot\frac{\frac{u}{\psi} - \tan(tk\psi)}{1 + \frac{u}{\psi}\tan(tk\psi)} = \psi\cdot\frac{\tan(\alpha) - \tan(tk\psi)}{1 + \tan(\alpha)\tan(tk\psi)}\ , \quad \text{where } \tan(\alpha) = \tfrac{u}{\psi}$$

$$v = \psi\tan(\alpha - tk\psi) = \sqrt{\tfrac{g}{k}}\tan\left(\alpha - tk\sqrt{\tfrac{g}{k}}\right) = \sqrt{\tfrac{g}{k}}\tan\left(\alpha - t\sqrt{gk}\right)\ , \quad \text{where } \alpha = \tan^{-1}\left(u\sqrt{\tfrac{k}{g}}\right)$$

x in terms of v: $\boxed{x = \frac{1}{2k}\ln\left|\frac{g+ku^2}{g+kv^2}\right|}$

$$v\cdot\frac{dv}{dx} = -g - kv^2\ , \quad \frac{dx}{dv} = \frac{v}{-g-kv^2}\ ,$$

$$x = \int_u^v \frac{v}{-g-kv^2}\,dv = -\frac{1}{2k}\int_{w(u)}^{w(v)} \frac{dw}{w}\ , \quad \text{where } w = \frac{g}{k} + v^2\ , \ dw = 2v\,dv$$

$$x = -\frac{1}{2k}\left[\ln\left|\frac{g}{k} + v^2\right|\right]_u^v = \frac{1}{2k}\ln\left|\frac{g+ku^2}{g+kv^2}\right|$$

x in terms of t: $\boxed{x = \frac{1}{k}\cdot\ln\left|\frac{\cos\left(\alpha - tk\sqrt{\frac{g}{k}}\right)}{\cos\alpha}\right|\ , \quad \text{where } \alpha = \tan^{-1}\left(u\sqrt{\tfrac{k}{g}}\right)}$

$$x = \int_0^t v\,dt = \int_0^t \sqrt{\tfrac{g}{k}}\tan\left(\alpha - tk\sqrt{\tfrac{g}{k}}\right)\,dt\ , \quad \text{where } \alpha = \tan^{-1}\left(u\sqrt{\tfrac{k}{g}}\right)$$

$$= \frac{1}{k}\int_{w(0)}^{w(t)} \tan(w)\ dw\ , \quad \text{where } w = \alpha - tk\sqrt{\tfrac{g}{k}}\ , \ dw = \sqrt{gk}\,dt$$

$$= \frac{1}{k}\left[\ln\left|\cos\left(\alpha - tk\sqrt{\tfrac{g}{k}}\right)\right|\right]_0^t$$

$$= \frac{1}{k}\cdot\ln\left|\frac{\cos\left(\alpha - tk\sqrt{\frac{g}{k}}\right)}{\cos\alpha}\right|$$

Max height H at T: $\boxed{H = \frac{1}{2k}\ln\left|1 + \frac{ku^2}{g}\right|}$ and $\boxed{T = \frac{\alpha}{\sqrt{gk}}\ , \quad \text{where } \alpha = \tan^{-1}\left(u\sqrt{\tfrac{k}{g}}\right)}$

At max, $\dfrac{dx}{dt} = v = 0$ and $x = H = \dfrac{1}{2k}\ln\left|\dfrac{g+ku^2}{g+kv^2}\right| = \dfrac{1}{2k}\ln\left|\dfrac{g+ku^2}{g}\right| = \dfrac{1}{2k}\ln\left|1 + \dfrac{ku^2}{g}\right|$

$t = T$ and $v = \sqrt{\tfrac{g}{k}}\tan\left(\alpha - T\sqrt{gk}\right) = 0$

$$\alpha - T\sqrt{gk} = 0\ , \quad T = \frac{\alpha}{\sqrt{gk}}$$

Special Case: When $u = \sqrt{\tfrac{g}{k}}$, $\alpha = \tfrac{\pi}{4}$, $H = \dfrac{1}{2k}\ln 2$ and $T = \dfrac{\pi}{4\sqrt{gk}} = \dfrac{\pi}{4ku} = \dfrac{u\pi}{4g}$

Free Fall: Based on the above formulae, but with $u = 0$ and the x-axis is downward,

$$m\ddot{x} = W + R = mg - mkv^2 , \qquad \therefore \quad \boxed{\ddot{x} = g - kv^2}$$

Given a free fall with no initial speed, i.e. when $t = 0,\ x = 0,\ \dot{x} = u = 0,\ \alpha = \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) = 0,$

v in terms of t: $\boxed{v = \sqrt{\dfrac{g}{k}} \cdot \left(\dfrac{1 - e^{-2\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right)}$

$$\frac{dv}{dt} = \ddot{x} = g - kv^2 , \quad \frac{dt}{dv} = \frac{1}{g - kv^2} , \quad t = \int_0^v \frac{dv}{g - kv^2} = \frac{1}{2\sqrt{gk}} \ln\left|\frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}\right| = \frac{1}{2\sqrt{gk}} \ln\left|\frac{g + v\sqrt{gk}}{g - v\sqrt{gk}}\right|$$

$$e^{2\sqrt{gk}\,t} = \frac{g + v\sqrt{gk}}{g - v\sqrt{gk}} , \quad g - v\sqrt{gk} = \left(g + v\sqrt{gk}\right) \cdot e^{-2\sqrt{gk}\,t}$$

$$g - g \cdot e^{-2\sqrt{gk}\,t} = v\sqrt{gk} + v\sqrt{gk} \cdot e^{-2\sqrt{gk}\,t} , \quad v = \sqrt{\frac{g}{k}} \cdot \left(\frac{1 - e^{-2\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right)$$

Terminal Velocity: $v_T = \lim\limits_{t \to +\infty} v = \lim\limits_{t \to +\infty} \sqrt{\dfrac{g}{k}} \cdot \left(\dfrac{1 - e^{-2\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right) = \sqrt{\dfrac{g}{k}}$

Also, $\ddot{x} = g - kv_T^2 = 0 , \quad \therefore \boxed{v_T = \sqrt{\dfrac{g}{k}}}$

x in terms of v: $\boxed{x = -\dfrac{1}{2k} \ln\left|1 - \dfrac{k}{g} v^2\right|}$

$$v \cdot \frac{dv}{dx} = g - kv^2 , \quad \frac{dx}{dv} = \frac{v}{g - kv^2} ,$$

$$x = \int_0^v \frac{v}{g - kv^2}\, dv = -\frac{1}{2k} \int_{w(0)}^{w(v)} \frac{dw}{w} , \quad \text{where } w = \frac{g}{k} - v^2 ,\ dw = -2v\, dv$$

$$x = -\frac{1}{2k}\left[\ln\left|\frac{g}{k} - v^2\right|\right]_0^v = -\frac{1}{2k}\left(\ln\left|\frac{g}{k} - v^2\right| - \ln\left|\frac{g}{k}\right|\right) = -\frac{1}{2k} \ln\left|1 - \frac{k}{g} v^2\right|$$

x in terms of t: $\boxed{x = \sqrt{\dfrac{g}{k}}\left[t + \dfrac{1}{\sqrt{gk}} \ln\left|\dfrac{1 + e^{-2\sqrt{gk}\,t}}{2}\right|\right]}$

$$x = -\frac{1}{2k} \ln\left|1 - \frac{k}{g} v^2\right| = \frac{1}{2k} \ln\left|\left(1 + \sqrt{\frac{k}{g}}\, v\right)\left(1 - \sqrt{\frac{k}{g}}\, v\right)\right|$$

$$= -\frac{1}{2k} \ln\left|\left(1 + \frac{1 - e^{-2\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right)\left(1 - \frac{1 - e^{-2\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right)\right| = -\frac{1}{2k} \ln\left|\left(\frac{2}{1 + e^{-2\sqrt{gk}\,t}}\right)\left(\frac{2\, e^{-2\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right)\right|$$

$$= -\frac{1}{2k} \ln\left|\left(\frac{2\, e^{-\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right)^2\right| = -\frac{1}{\cancel{2}k} \cdot \cancel{2} \ln\left|\frac{2\, e^{-\sqrt{gk}\,t}}{1 + e^{-2\sqrt{gk}\,t}}\right| = \frac{1}{k} \cdot \ln\left|\frac{1 + e^{-2\sqrt{gk}\,t}}{2\, e^{-\sqrt{gk}\,t}}\right|$$

$$= \frac{1}{k} \cdot \left(\ln\left|\frac{1 + e^{-2\sqrt{gk}\,t}}{2}\right| - \ln\left|e^{-\sqrt{gk}\,t}\right|\right) = \frac{1}{k} \cdot \left(\ln\left|\frac{1 + e^{-2\sqrt{gk}\,t}}{2}\right| + \sqrt{gk}\, t\right)$$

$$= \sqrt{\frac{g}{k}}\, t + \frac{1}{k} \ln\left|\frac{1 + e^{-2\sqrt{gk}\,t}}{2}\right| = \sqrt{\frac{g}{k}}\left[t + \frac{1}{\sqrt{gk}} \ln\left|\frac{1 + e^{-2\sqrt{gk}\,t}}{2}\right|\right]$$