Projectile Motion

Basic Equations: Horizontal Force $F_H = m\ddot{x} = 0$, Vertical Force $F_V = m\ddot{y} = -mg$. So $\boxed{\ddot{x} = 0, \quad \ddot{y} = -g}$.

Let $U = v_0$ and θ be the angle of its inclination, then $\boxed{\dot{x} = U\cos\theta, \quad \dot{y} = U\sin\theta}$.

$$\therefore \boxed{x = Ut\cos\theta, \quad y = Ut\sin\theta - \frac{1}{2}gt^2}$$

Cartesian: $t = \dfrac{x}{U\cos\theta}$, $y = U\sin\theta \cdot \dfrac{x}{U\cos\theta} - \dfrac{1}{2}g \cdot \dfrac{x^2}{U^2\cos^2\theta}$, $\boxed{y = \tan\theta \cdot x - \dfrac{gx^2}{2U^2\cos^2\theta}}$

Maximum Reach: When the object hits the ground, $y = \tan\theta \cdot x - \dfrac{gx^2}{2U^2\cos^2\theta} = 0$

$$x = \frac{U^2}{g} \cdot 2\cos^2\theta \cdot \tan\theta = \frac{U^2}{g} \cdot 2\sin\theta\cos\theta = \frac{U^2}{g} \cdot \sin 2\theta$$

When $\theta = \dfrac{\pi}{4}$, $\sin 2\theta = 1$ and x is at its maximum: $\boxed{x_{max} = \dfrac{U^2}{g}, \quad \text{when } \theta = \dfrac{\pi}{4}}$

Maximum Height: $\dot{y} = U\sin\theta - gT = 0$, $\therefore \boxed{T = \dfrac{U\sin\theta}{g}}$. At that time ...

$$y_{max} = UT\sin\theta - \frac{1}{2}gT^2 = U\sin\theta \cdot \frac{U\sin\theta}{g} - \frac{1}{2}g \cdot \frac{U^2\sin^2\theta}{g^2}, \quad \boxed{H = \frac{U^2\sin^2\theta}{2g}}$$

Resistant Medium: $\ddot{x} = -k\dot{x}$, $\ddot{y} = -g - k\dot{y}$

Let U be the initial speed, then $U_x = U\cos\theta$ and $U_y = U\sin\theta$.

Vertical: Based on formulae from Motion of Resistance Proportional to Speed:

$$y = -\frac{g}{k}t + \left(\frac{g + kU\sin\theta}{k^2}\right)\left(1 - e^{-kt}\right) \quad \ldots (1)$$

$$\dot{y} = -\frac{g}{k} + \frac{g + kU\sin\theta}{k} \cdot e^{-kt} \quad \ldots (2)$$

Horizontal: Take the above formulae and substitute $g = 0$.

$$x = \left(\frac{kU_x}{k^2}\right)\left(1 - e^{-kt}\right) = \frac{U\cos\theta}{k}(1 - e^{-kt}) \quad \ldots (3)$$

$$\dot{x} = \frac{kU_x}{k} \cdot e^{-kt} = U\cos\theta \cdot e^{-kt} \quad \ldots (4)$$

Cartesian:

From (3): $1 - e^{-kt} = \dfrac{kx}{U\cos\theta} \quad \ldots (5)$

$$e^{-kt} = 1 - \frac{kx}{U\cos\theta}\,, \quad -kt = \ln\left(1 - \frac{kx}{U\cos\theta}\right)$$

$$t = -\frac{1}{k}\ln\left(1 - \frac{kx}{U\cos\theta}\right) \quad \ldots (6)$$

Sub (5),(6) into (1): $y = -\dfrac{g}{k}t + \left(\dfrac{g + kU\sin\theta}{k^2}\right)\left(1 - e^{-kt}\right)$

$$= \frac{g}{k}\cdot\frac{1}{k}\ln\left(1 - \frac{kx}{U\cos\theta}\right) + \left(\frac{g + kU\sin\theta}{k^2}\right)\cdot\frac{kx}{U\cos\theta}$$

$$\boxed{y = \frac{g}{k^2}\ln\left(1 - \frac{kx}{U\cos\theta}\right) + \left(\frac{g}{kU\cos\theta} + \tan\theta\right)x}$$

Eventually:

From (3): $\displaystyle\lim_{t\to\infty} x = \lim_{t\to\infty}\left[\frac{U\cos\theta}{k}(1 - e^{-kt})\right] = \frac{U\cos\theta}{k}$

From (1): $\displaystyle\lim_{t\to\infty} y = \lim_{t\to\infty}\left[-\frac{g}{k}t + \left(\frac{g + kU\sin\theta}{k^2}\right)\left(1 - e^{-kt}\right)\right] = -\infty$

From (2): $\displaystyle\lim_{t\to\infty} \dot{y} = \lim_{t\to\infty}\left[-\frac{g}{k} + \frac{g + kU\sin\theta}{k}\cdot e^{-kt}\right] = -\frac{g}{k}$

The object eventually falls at a terminal speed of $\dfrac{g}{k}$

vertically along $x_T = \dfrac{U\cos\theta}{k}$, so $0 < x < x_T$ at all time.

Drag or No Drag: Compare drag formula y_d with the no-drag formula $y_{nd} = \tan\theta\cdot x - \dfrac{gx^2}{2U^2\cos^2\theta}$

$$y_d - y_{nd} = \left[\frac{g}{k^2}\ln\left(1 - \frac{kx}{U\cos\theta}\right) + \frac{gx}{kU\cos\theta} + \tan\theta\cdot x\right] - \left[\tan\theta\cdot x - \frac{gx^2}{2U^2\cos^2\theta}\right]$$

$$= \frac{g}{k^2}\left[\ln\left(1 - \frac{kx}{U\cos\theta}\right) + \frac{kx}{U\cos\theta} + \frac{k^2x^2}{2U^2\cos^2\theta}\right]$$

$$= \frac{g}{k^2}\left[\ln\left(1 - \frac{x}{x_T}\right) + \frac{x}{x_T} + \frac{1}{2}\left(\frac{x}{x_T}\right)^2\right] < 0$$

$$= \frac{g}{k^2}\left[\ln(1-\lambda) + \lambda + \frac{\lambda^2}{2}\right] < 0 \quad \text{where } \lambda = \frac{x}{x_T}\,, \quad \text{and } 0 < \lambda < 1\,.$$

Given $\dfrac{g}{k^2} > 0$, let us consider $f(\lambda) = \ln(1-\lambda) + \lambda + \dfrac{\lambda^2}{2}$.

$f(0) = 0$ and $f'(\lambda) = \dfrac{-1}{1-\lambda} + 1 + \lambda = \dfrac{-\lambda^2}{1-\lambda} < 0$.

$\therefore f(\lambda) < 0$ for $\lambda \in (0,1)$. So $y_d - y_{nd} = \dfrac{g}{k^2}f(\lambda) < 0$.

$\therefore \boxed{y_d < y_{nd}}$ for $0 < x < x_T$ (domain of y_d)

Minute Resistance: When the resistance is small ($k \to 0$), the above should reduce to the "non-drag" case.

Cartesian: Given $\ln(1+n) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{n^r}{r}$, $\ln\left(1 - \frac{kx}{U\cos\theta}\right) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{(-\frac{kx}{U\cos\theta})^r}{r} = -\sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{kx}{U\cos\theta}\right)^r$

$$y = \lim_{k\to 0}\left[\frac{g}{k^2}\ln\left(1 - \frac{kx}{U\cos\theta}\right) + \left(\frac{g}{kU\cos\theta} + \tan\theta\right)x\right]$$

$$= \lim_{k\to 0}\left[-\frac{g}{k^2}\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{kx}{U\cos\theta}\right)^r + \frac{gx}{kU\cos\theta} + \tan\theta\cdot x\right]$$

$$= \lim_{k\to 0}\left[-\frac{g}{k^2}\left(\frac{kx}{U\cos\theta} + \frac{1}{2}\left(\frac{kx}{U\cos\theta}\right)^2 + \sum_{r=3}^{\infty}\frac{1}{r}\left(\frac{kx}{U\cos\theta}\right)^r\right) + \frac{gx}{kU\cos\theta} + \tan\theta\cdot x\right]$$

$$= \lim_{k\to 0}\left[-\frac{gx}{kU\cos\theta} - \frac{g}{2k^2}\left(\frac{kx}{U\cos\theta}\right)^2 - \frac{g}{k^2}\sum_{r=3}^{\infty}\frac{1}{r}\left(\frac{kx}{U\cos\theta}\right)^r + \frac{gx}{kU\cos\theta} + \tan\theta\cdot x\right]$$

$$= \lim_{k\to 0}\left[-\frac{g}{2}\left(\frac{x}{U\cos\theta}\right)^2 - g\sum_{r=3}^{\infty}\frac{1}{k^2 r}\left(\frac{kx}{U\cos\theta}\right)^r + \tan\theta\cdot x\right]$$

$$= -\lim_{k\to 0}\left[\frac{g}{2}\left(\frac{x}{U\cos\theta}\right)^2\right] - \lim_{k\to 0}\left[g\sum_{r=3}^{\infty}\frac{1}{k^2 r}\left(\frac{kx}{U\cos\theta}\right)^r\right] + \lim_{k\to 0}(\tan\theta\cdot x)$$

$\therefore \quad y = \tan\theta\cdot x - \frac{gx^2}{2U^2\cos^2\theta}$ same as the non-drag equation

$\ddot{x}$: $\ddot{x} = \lim_{k\to 0}(-k\dot{x}) = 0 = \ddot{x}_{nd}$

$\dot{x}$: From (4): $\dot{x} = \lim_{k\to 0}(U\cos\theta\cdot e^{-kt}) = U\cos\theta\cdot e^0 = U\cos\theta = \dot{x}_{nd}$

x: Given $e^n = \sum_{r=0}^{\infty}\frac{n^r}{r!}$, $\lim_{k\to 0}\frac{1-e^{-kt}}{k} = \lim_{k\to 0}\frac{1}{k}\left(1 - \sum_{r=0}^{\infty}\frac{(-kt)^r}{r!}\right) = \lim_{k\to 0}\frac{1}{k}\left[1 - \left(1 - kt + \sum_{r=2}^{\infty}\frac{(-kt)^r}{r!}\right)\right]$

$$= \lim_{k\to 0}\left(t - \frac{1}{k}\sum_{r=2}^{\infty}\frac{(-kt)^r}{r!}\right) = \lim_{k\to 0} t - \lim_{k\to 0}\sum_{r=2}^{\infty}\frac{(-kt)^r}{kr!} = t - 0 = t\,.$$

From (3): $x = \lim_{k\to 0}\frac{U\cos\theta}{k}(1 - e^{-kt}) = U\cos\theta\cdot\lim_{k\to 0}\left(\frac{1-e^{-kt}}{k}\right) = Ut\cos\theta = x_{nd}$

$\ddot{y}$: $\ddot{y} = \lim_{k\to 0}(-g - k\dot{y}) = -g = \ddot{y}_{nd}$

$\dot{y}$: From (2): $\dot{y} = \lim_{k\to 0}\left(-\frac{g}{k} + \frac{g + kU\sin\theta}{k}\cdot e^{-kt}\right) = \lim_{k\to 0}\left(-\frac{g}{k} + \frac{g}{k}\cdot e^{-kt} + U\sin\theta\cdot e^{-kt}\right)$

$$= -g\lim_{k\to 0}\left(\frac{1-e^{-kt}}{k}\right) + \lim_{k\to 0}\left(U\sin\theta\cdot e^{-kt}\right) = U\sin\theta - gt = \dot{y}_{nd}$$

y: First, let us find $\lim_{k\to 0}\frac{1 - e^{-kt} - kt}{k^2} = \lim_{k\to 0}\frac{1}{k^2}\left(1 - \sum_{r=0}^{\infty}\frac{(-kt)^r}{r!} - kt\right)$

$$= \lim_{k\to 0}\frac{1}{k^2}\left[1 - \left(1 - kt + \frac{k^2t^2}{2!} + \sum_{r=3}^{\infty}\frac{(-kt)^r}{r!}\right) - kt\right]$$

$$= \lim_{k\to 0}\left(-\frac{t^2}{2} + \sum_{r=3}^{\infty}\frac{(-kt)^r}{k^2 r!}\right) = -\frac{t^2}{2} + \lim_{k\to 0}\sum_{r=3}^{\infty}\frac{(-kt)^r}{k^2 r!} = -\frac{t^2}{2}$$

From (1): $y = \lim_{k\to 0}\left[-\frac{g}{k}t + \left(\frac{g + kU\sin\theta}{k^2}\right)(1 - e^{-kt})\right] = \lim_{k\to 0}\left[-\frac{g}{k}t + \left(\frac{g}{k} + U\sin\theta\right)\left(\frac{1-e^{-kt}}{k}\right)\right]$

$$= \lim_{k\to 0}\left[U\sin\theta\left(\frac{1-e^{-kt}}{k}\right) + \frac{g}{k}\left(\frac{1-e^{-kt}}{k} - t\right)\right] = U\sin\theta\cdot\lim_{k\to 0}\left(\frac{1-e^{-kt}}{k}\right) + g\lim_{k\to 0}\left(\frac{1-e^{-kt}-kt}{k^2}\right)$$

$$= Ut\sin\theta - \frac{1}{2}gt^2 = y_{nd}$$