Conical Pendulum

Hang off a point: An object of mass m suspended by a string of length l is performing uniform circular motion with angular velocity ω. The inclination of the string and the vertical axis is θ. There is tension T on the string. The plane of circular motion is h under the suspension point A.

$$\ddot{x} = r\omega^2 \,, \quad \text{where} \quad r = l\sin\theta \,.$$

$$F_N = T\sin\theta = m\ddot{x} = mr\omega^2 = ml\sin\theta\cdot\omega^2 \,, \quad \boxed{T = ml\omega^2} \,.$$

To balance the weight: $T\cos\theta = mg$, $ml\omega^2\cos\theta = mg$, $\boxed{\cos\theta = \dfrac{g}{l\omega^2}}$.

$$h = l\cos\theta = \frac{g}{\omega^2} \,, \quad \boxed{h = \frac{g}{\omega^2}} \,.$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg} = \frac{r^2\omega^2}{rg} = \frac{v^2}{rg} \,, \quad \boxed{\tan\theta = \frac{v^2}{rg}} \,.$$

Frequency $F = \dfrac{\omega}{2\pi} = \dfrac{\sqrt{\omega^2}}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{h}}$, $\boxed{f = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{h}}}$.

Period $T = \dfrac{1}{f}$, $\boxed{T = 2\pi\sqrt{\dfrac{h}{g}}}$.

Hang off a disc: An object of mass m suspended by a string of length l hanging off the rim of a disc of radius R is performing uniform circular motion with angular velocity ω. The inclination of the string and the vertical axis is θ. There is tension T on the string. The plane of motion is h under the disc.

Radius of the circular motion $r = R + l\sin\theta$, $\ddot{x} = r\omega^2 = (R + l\sin\theta)\omega^2$, $h = l\cos\theta$.

$$F_N = T\sin\theta = m\ddot{x} = mr\omega^2 = m(R + l\sin\theta)\omega^2 \,, \quad \boxed{T = m\left(\frac{R}{\sin\theta} + l\right)\omega^2} \,.$$

To balance the weight: $mg = T\cos\theta = m\left(\dfrac{R}{\sin\theta} + l\right)\omega^2\cos\theta$, $\dfrac{g}{\omega^2} = \dfrac{R}{\tan\theta} + l\cos\theta = \dfrac{R}{\tan\theta} + h$

$$\boxed{\tan\theta = \frac{R\omega^2}{g - h\omega^2}} \,, \quad \boxed{\omega = \sqrt{\frac{g\tan\theta}{R + l\sin\theta}}} \,, \quad \boxed{h = \frac{g}{\omega^2} - \frac{R}{\tan\theta}} \,,$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg} = \frac{r^2\omega^2}{rg} = \frac{v^2}{rg} \,, \quad \boxed{\tan\theta = \frac{v^2}{(R + l\sin\theta)g}} \,, \quad \boxed{v^2 = \tan\theta(R + l\sin\theta)g} \,.$$

Frequency $F = \dfrac{\omega}{2\pi}$, $\boxed{f = \dfrac{1}{2\pi}\sqrt{\dfrac{g\tan\theta}{R + l\sin\theta}}}$.

Period $T = \dfrac{1}{f}$, $\boxed{T = 2\pi\sqrt{\dfrac{R + l\sin\theta}{g\tan\theta}}}$.

Note: When $R = 0$, the disc becomes a point, so the above is reduced to the case of “hanging off a point”.