Banked Track

Configuration: A car of mass m on a curved banked track of radius r is in uniform circular motion. The inclination of the track to the horizontal is θ . The track supports the car by a normal force of N perpendicular to the surface, and a sideways frictional force of F towards the inner rim (lower side) of the track.

Horizontal: $N\sin\theta + F\cos\theta = mr\omega^2 = m\dfrac{v^2}{r} \quad \ldots (1)$

Vertical: $N\cos\theta - F\sin\theta = mg \quad \ldots (2)$

$(1)\cos\theta - (2)\sin\theta :\quad F\,(\cos^2\theta + \sin^2\theta) = m\dfrac{v^2}{r}\cos\theta - mg\sin\theta\,,\quad F = mg\cos\theta\left(\dfrac{v^2}{rg} - \tan\theta\right) \quad \ldots (3)$

$(1)\sin\theta + (2)\cos\theta :\quad N\,(\sin^2\theta + \cos^2\theta) = m\dfrac{v^2}{r}\sin\theta + mg\cos\theta\,,\quad N = mg\cos\theta\left(\dfrac{v^2}{rg}\tan\theta + 1\right) \quad \ldots (4)$

Parallel to track: From (3): $F + mg\sin\theta = m\dfrac{v^2}{r}\cos\theta$

Perpendicular: From (4): $N - mg\cos\theta = m\dfrac{v^2}{r}\sin\theta$

Friction F: In (3), m, g, θ and r are constants, so there is a critical speed $v = V$ to make $F = 0$ (no sideways friction).

$$F = mg\cos\theta\left(\frac{V^2}{rg} - \tan\theta\right) = 0\,,\quad \boxed{\tan\theta = \frac{V^2}{rg}}\,,\quad \boxed{V = \sqrt{rg\tan\theta}}\,.$$

The road is designed such that friction is minimal at the "usual" speed, which is V. If the road is designed for higher speed, the inclination should be higher. The track is to be banked inwards by $\theta = \tan^{-1}\left(\frac{V^2}{rg}\right)$.

$$\text{From (3): } F = mg\cos\theta\left(\frac{v^2}{rg} - \tan\theta\right) = mg\cos\theta\left(\frac{v^2}{rg} - \frac{V^2}{rg}\right)\,,\quad \boxed{F = m\frac{(v^2 - V^2)}{r}\cos\theta} \quad \ldots (5)$$

For a small θ, $\boxed{F \simeq m\dfrac{(v^2 - V^2)}{r}} \quad \ldots (6)$

$\dfrac{V^2}{rg} = \tan\theta \simeq \sin\theta = \dfrac{h}{d}$, where h is the outer rim height and d is the width of the track. So $\boxed{h \simeq \dfrac{V^2 d}{rg}}$.

Inclination θ: On an unlaned track, the car may run on any part of the track. From (5), when the car is running below the critical speed, $v < V$, F becomes negative, pushing the car outwards while it is drifting inwards. To balance the forces, the critical speed V should be lowered as well to minimise $v^2 - V^2$ (and therefore friction). A lower V means a smaller θ (as $\tan\theta = \frac{V^2}{rg}$). That is why the track is less inclined at the inner rim.

On the other hand, at high speed, $v > V$ and F becomes positive, pushing the car inwards while it is drifting outwards. So V needs to be higher to get F back to neutral (close to zero). A higher V means a larger θ, so the track is more inclined at the outer rim (and (6) no longer applies).

A well designed track should allow a free skidding car (when out of control) to stay within the track without much help from friction (as the tyres are most likely not gripping). The steep inclination at the outer rim should push the car back in when it is too fast for the curvature. When the car slows down, the flattened inner rim should steadily reduce its sideway motion until it eventually grips again and stops safely.

Also from (5), if a track is designed for a constant V, at the section of sharp turn, r becomes small, and therefore requires a small $\cos\theta$ to balance the friction. A smaller cosine value means a larger θ . That is why the track is more inclined at sharp turn (e.g. a skeleton sled sliding down the curved track at high speeds).

Co-efficient μ: The friction F is proportional to the normal force N up to a limit before the object slips. The upper limit of $\left|\frac{F}{N}\right|$ is μ, which is called the co-efficient of friction. When $\left|\frac{F}{N}\right|$ is larger than μ , the car starts to slip.

$\because N > 0\,, \quad \therefore |F| \leq \mu N\,, \quad -\mu N \leq F \leq \mu N\,. \quad \boxed{F_{max} = \mu N, \quad F_{min} = -\mu N} \quad \ldots (7)\,.$

$$(3) \div (4): \quad \frac{F}{N} = \frac{\frac{v^2}{rg} - \tan\theta}{\frac{v^2}{rg}\tan\theta + 1}\,, \quad \boxed{\frac{F}{N} = \frac{v^2 - rg\tan\theta}{v^2\tan\theta + rg}} \quad \ldots (8)$$

When $F = 0\,, \quad V^2 - rg\tan\theta = 0\,, \quad \tan\theta = \dfrac{V^2}{rg}\,,$ which reconfirm the previous result.

When v is at its maximum, F is at its maximum too, and the car's weight is on the outer wheels.

From (7), (8): $\mu = \dfrac{F_{max}}{N} = \dfrac{v_{max}^2 - rg\tan\theta}{v_{max}^2\tan\theta + rg}\,, \quad \mu v_{max}^2\tan\theta + \mu rg = v_{max}^2 - rg\tan\theta\,,$

$v_{max}^2(1 - \mu\tan\theta) = rg(\tan\theta + \mu)\,, \quad \boxed{v_{max}^2 = rg \cdot \dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}}\,.$ Both sides need to be positive, so $\mu < \dfrac{1}{\tan\theta}\,.$

The closer μ is to $\frac{1}{\tan\theta}$, the larger v_{max} can be. When $\mu >= \dfrac{1}{\tan\theta}$, the car will never slip even at high speed.

When v is at its minimum, F is at its minimum too, and the car's weight is on the inner wheels.

From (7), (8): $\mu = -\dfrac{F_{min}}{N} = \dfrac{rg\tan\theta - v_{min}^2}{v_{min}^2\tan\theta + rg}\,, \quad \mu v_{min}^2\tan\theta + \mu rg = rg\tan\theta - v_{min}^2\,,$

$v_{min}^2(1 + \mu\tan\theta) = rg(\tan\theta - \mu)\,, \quad \boxed{v_{min}^2 = rg \cdot \dfrac{\tan\theta - \mu}{1 + \mu\tan\theta}}\,.$ Both sides need to be positive, so $\mu < \tan\theta\,.$

The closer μ is to $\tan\theta$, the smaller v_{min} can be. When $\mu >= \tan\theta$, the car will never slip even when it stops.

In civil engineering, when a road is designed, r is determined by the geography and the average speed V is determined by the traffic (e.g. freeway, local street). The inclination can therefore be determined from $\theta = \tan^{-1}\left(\frac{V^2}{rg}\right)$. While μ is largely determined by the tyres of vehicles and the road surface material and condition, it needs to satisfy $\mu >= \tan\theta$ so the car can stop when required without rolling down the banked track. Also, v_{max} needs to be high enough to prevent reckless drivers from killing themselves and others, literally along the way. Therefore, θ needs to be engineered to make sure all these criteria are met. If a solution for θ cannot be found, that means r needs to be enlarged (e.g. by bridging around a hairpin) or reduce the recommended speed (e.g. using signs and speed cameras).