Mathematical Induction – Trigonometry

Example $P(n): \quad S_n = \sum_{r=1}^{n} \sin(2r-1)\theta = \dfrac{1-\cos 2n\theta}{2\sin\theta}$

$$P(1): \quad LHS = \sum_{r=1}^{1} \sin(2r-1)\theta = \sin\theta\,, \quad RHS = \frac{1-\cos 2\theta}{2\sin\theta} = \frac{1-1+2\sin^2\theta}{2\sin\theta} = \sin\theta$$

$LHS = RHS$, $\therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $S_k = \sum_{r=1}^{k} \sin(2r-1)\theta = \dfrac{1-\cos 2k\theta}{2\sin\theta}$

$$P(k+1): \quad S_{k+1} = \sum_{r=1}^{k+1} \sin(2r-1)\theta = \sum_{r=1}^{k} \sin(2r-1)\theta + \sin[2(k+1)-1]\theta$$

$$= \frac{1-\cos 2k\theta}{2\sin\theta} + \sin(2k+1)\theta = \frac{1-\cos 2k\theta + 2\sin\theta\ \sin(2k\theta+\theta)}{2\sin\theta}$$

$$= \frac{1-\cos 2k\theta - \cos(2k\theta+2\theta) + \cos 2k\theta}{2\sin\theta} = \frac{1-\cos[(2(k+1)\theta]}{2\sin\theta}$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad S_n = \sum_{r=1}^{n} 2^{r-1}\tan(2^{r-1}\theta) = \cot\theta - 2^n\cot(2^n\theta)$ for $n \in +\mathbb{Z}$.

$$P(1): \quad LHS = \sum_{r=1}^{1} 2^{r-1}\tan(2^{r-1}\theta) = \tan\theta\,,$$

$$RHS = \cot\theta - 2\cot(2\,\theta) = \frac{\cos\theta}{\sin\theta} - \frac{2\cdot\cos(2\,\theta)}{\sin(2\,\theta)} = \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\not{2}\cdot(1-2\sin^2\theta)}{\not{2}\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta - (1-2\sin^2\theta)}{\sin\theta\cos\theta} = \frac{2\sin^2\theta - (1-\cos^2\theta)}{\sin\theta\cos\theta} = \frac{\sin^2\theta}{\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta\,,$$

$LHS = RHS$, $\therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $S_k = \sum_{r=1}^{k} 2^{r-1}\tan(2^{r-1}\theta) = \cot\theta - 2^k\cot(2^k\theta)$

$$P(k+1): \quad S_{k+1} = \sum_{r=1}^{k+1} 2^{r-1}\tan(2^{r-1}\theta) = \sum_{r=1}^{k} 2^{r-1}\tan(2^{r-1}\theta) + 2^{(k+1)-1}\tan(2^{(k+1)-1}\theta)$$

$$= \cot\theta - 2^k\cot(2^k\theta) + 2^k\tan(2^k\theta) = \cot\theta - 2^k[\cot(2^k\theta) - \tan(2^k\theta)]$$

$$= \cot\theta - 2^k \cdot \frac{1-\tan^2(2^k\theta)}{\tan(2^k\theta)} = \cot\theta - 2^{k+1}\cdot\frac{1-\tan^2(2^k\theta)}{2\tan(2^k\theta)}$$

$$= \cot\theta - 2^{k+1}\cdot\frac{1}{\tan(2\cdot 2^k\theta)} = \cot\theta - 2^{k+1}\cot(2^{k+1}\theta)$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad S_n = \sum_{r=1}^{n} \sin r\theta = \dfrac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$ for $n \in +\mathbb{Z}$.

$$P(1): \quad LHS = \sum_{r=1}^{1} \sin r\theta = \sin\theta\,,$$

$$RHS = \frac{\sin\frac{\theta}{2}\sin\frac{(1+1)\theta}{2}}{\sin\frac{\theta}{2}} = \sin\theta\,,$$

$LHS = RHS\,, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $S_k = \sum_{r=1}^{k} \sin r\theta = \dfrac{\sin\frac{k\theta}{2}\sin\frac{(k+1)\theta}{2}}{\sin\frac{\theta}{2}}$

$$P(k+1): \quad S_{k+1} = \sum_{r=1}^{k+1} \sin r\theta = \sum_{r=1}^{k} \sin r\theta + \sin(k+1)\theta$$

$$= \frac{\sin\frac{k\theta}{2}\sin\frac{(k+1)\theta}{2}}{\sin\frac{\theta}{2}} + \sin(k+1)\theta$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left[\sin\frac{k\theta}{2}\sin\frac{(k+1)\theta}{2} + \sin\frac{\theta}{2}\sin(k+1)\theta\right]$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left[-\frac{1}{2}\left(\cos\frac{(2k+1)\theta}{2} - \cos\frac{\theta}{2}\right) - \frac{1}{2}\left(\cos\frac{(2k+3)\theta}{2} - \cos\frac{(2k+1)\theta}{2}\right)\right]$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left[\frac{1}{2}\left(\cos\frac{\theta}{2} - \cos\frac{(2k+3)\theta}{2}\right)\right]$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left[\frac{1}{2}\cdot 2\left(\sin\frac{(k+2)\theta}{2}\sin\frac{(k+1)\theta}{2}\right)\right]$$

$$= \frac{\sin\frac{(k+1)\theta}{2}\sin\frac{[(k+1)+1]\theta}{2}}{\sin\frac{\theta}{2}}$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.