Mathematical Induction – Polynomials, Products and Factorials

Example $P(n): \quad 2n-1 \,\Big|\, \frac{1}{2^n}\prod_{r=n+1}^{2n} r \,,\qquad \text{i.e. } \frac{1}{2^n}(n+1)(n+2)\dots(2n)$

$P(1): \quad \frac{1}{2^1}\prod_{r=1+1}^{2} r = 1\,,$ which is divisible by $2\cdot 1-1=1\,,\quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k\geq 1$, then $\frac{1}{2^k}\prod_{r=k+1}^{2k} r = P(2k-1)\,,\quad$ for some $P\in\mathbb{Z}$

$$P(k+1): \quad \frac{1}{2^{k+1}}\prod_{r=(k+1)+1}^{2(k+1)} r$$
$$=\frac{1}{2}\cdot\frac{1}{2^k}\prod_{r=k+2}^{2k+2} r$$
$$=\frac{1}{2}\cdot\frac{1}{2^k}\prod_{r=k+1}^{2k} r\cdot\frac{(2k+1)(2k+2)}{k+1}$$
$$=\frac{1}{\cancel{2}}\cdot P(2k-1)\cdot\frac{\cancel{2}(2k+1)\cancel{(k+1)}}{\cancel{k+1}}$$
$$=P(2k-1)\cdot[2(k+1)-1] = Q[2(k+1)-1]\,,\quad \text{where } Q=P(2k-1)$$

$\therefore$ By the PMI, $P(n)$ is true for $n\in +\mathbb{Z}$.

Example $P(n): \quad \sum_{r=1}^{n}(r^2+r+1)r! = (n+1)^2 n! - 1$

$P(1): \quad LHS = \sum_{r=1}^{1}(r^2+r+1)r! = 3 = (1+1)^2 1! - 1 = RHS\,,\quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k\geq 1$, then $\sum_{r=1}^{k}(r^2+r+1)r! = (k+1)^2 k! - 1$

$$P(k+1): \quad \sum_{r=1}^{k+1}(r^2+r+1)r! = \sum_{r=1}^{k}(r^2+r+1)r! + [(k+1)^2+(k+1)+1](k+1)!$$
$$=[(k+1)^2 k! - 1] + [(k+1)^2+(k+1)+1](k+1)k!$$
$$=[(k+1)+(k+1)^2+(k+1)+1](k+1)k! - 1$$
$$=[(k+1)^2+2(k+1)+1](k+1)! - 1$$
$$=[(k+1)+1]^2(k+1)! - 1$$

$\therefore$ By the PMI, $P(n)$ is true for $n\in +\mathbb{Z}$.