Mathematical Induction – Inequalities

Example $P(n): \quad 2^n > n \quad$ for all integers $n \geq 0$.

$P(0): \quad 2^0 = 1 > 0\,, \quad \therefore P(0)$ is true.

$P(1): \quad 2^1 = 2 > 1\,, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $2^k > k$

$P(k+1): \quad 2^{k+1} = 2 \cdot 2^k > 2k = k + k \geq k + 1\,, \quad \therefore 2^{k+1} > k+1$

$\therefore$ By the PMI, $P(n)$ is true for all integers $n \geq 0$.

Similarly $P(n): \quad 2^n > n^2 \quad$ for all integers $n \geq 5$.

$P(5): \quad 2^5 = 32 > 5^2 = 25\,, \quad \therefore P(5)$ is true.

$P(k): \quad 2^k > k^2\,, \quad$ for $k \geq 5$

$$
\begin{aligned}
P(k+1): \quad & 2^{k+1} = 2 \cdot 2^k > 2k^2 \\
& = k^2 + k \cdot k \geq k^2 + 5k \\
& = k^2 + 2k + 3k > k^2 + 2k + 1 \\
& = (k+1)^2\,, \quad \therefore 2^{k+1} > (k+1)^2
\end{aligned}
$$

$P(n): \quad 2^n > n^3 \quad$ for all integers $n \geq 10$.

$P(10): \quad 2^{10} = 1024 > 10^3 = 1000\,, \quad \therefore P(10)$ is true.

$P(k): \quad 2^k > k^3\,, \quad$ for $k \geq 10$

$$
\begin{aligned}
P(k+1): \quad & 2^{k+1} = 2 \cdot 2^k > 2k^3 \\
& = k^3 + k \cdot k^2 \geq k^3 + 10k^2 \\
& = k^3 + 3k^2 + 7k^2 > k^3 + 3k^2 + 7k \cdot 10 \\
& = k^3 + 3k^2 + 3k + 67k > k^3 + 3k^2 + 3k + 1 \\
& = (k+1)^3\,, \quad \therefore 2^{k+1} > (k+1)^3
\end{aligned}
$$

Aternatively $P(n): \quad 2^n > n^4 \quad$ for all integers $n \geq 17$.

$P(17): \quad 2^{17} = 131072 > 17^4 = 83521\,, \quad \therefore P(17)$ is true.

$P(k): \quad 2^k > k^4\,, \quad$ for $k \geq 17$

i.e. $2^k - k^4 > 0$

$$
\begin{aligned}
P(k+1): \quad & 2^{k+1} - (k+1)^4 \\
& = 2 \cdot 2^k - k^4 - 4k^3 - 6k^2 - 4k - 1 > 2 \cdot k^4 - k^4 - 4k^3 - 6k^2 - 4k - 1 \\
& = k^4 - 4k^3 - 6k^2 - 4k - 1 \\
& \geq 17 \cdot (k^3 - 4k^2 - 6k - 4) - 1 \\
& \geq 17 \cdot [17 \cdot (k^2 - 4k - 6) - 4] - 1 \\
& \geq 17 \cdot [17 \cdot \big(17 \cdot (k-4) - 6\big) - 4] - 1 \\
& \geq 17 \cdot [17 \cdot \big(17 \cdot (17-4) - 6\big) - 4] - 1 > 0\,, \quad \therefore 2^{k+1} > (k+1)^4
\end{aligned}
$$

Example $P(n): \quad 3^n > 2n + 5$ for all integers $n \geq 3$.

$P(3): \quad 3^3 = 27 > 2 \cdot 3 + 5 = 11 , \quad \therefore P(3)$ is true.

If $P(k)$ is true up to some integer $k \geq 3$, then $3^k > 2k + 5$

$$P(k+1): \quad 3^{k+1} = 3 \cdot 3^k > 3 \cdot (2k+5)$$
$$= 6k + 15 > 2k + 7$$
$$= 2(k+1) + 5 , \quad \therefore 3^{k+1} > 2(k+1) + 5$$

$\therefore$ By the PMI, $P(n)$ is true for all integers $n \geq 3$.

Example $P(n): \quad n! \geq 2^{n-1}$ for all integers $n \geq 1$.

$P(1): \quad 1! = 1 \geq 2^{1-1} = 1 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $k! \geq 2^{k-1}$

$$P(k+1): \quad (k+1)! = (k+1)k! \geq (k+1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^{(k+1)-1}$$

$\therefore$ By the PMI, $P(n)$ is true for all integers $n \geq 1$.

Example a) If $a_1 > 1$ and $a_2 > 1$, then $a_1 a_2 - a_1 - a_2 + 1 > 0$

Proof: $a_1 - 1 > 0 , \quad a_2 - 1 > 0 , \quad (a_1 - 1)(a_2 - 1) > 0$

$\therefore \quad a_1 a_2 - a_1 - a_2 + 1 > 0$

b) $P(n): \quad \displaystyle\prod_{r=1}^{n} a_r > \sum_{r=1}^{n} a_r + 1 - n$ for all integers $n \geq 2$, where $a_r > 1$.

Let $\displaystyle P_n = \prod_{r=1}^{n} a_r$, and $\displaystyle S_n = \sum_{r=1}^{n} a_r$

$P(n): \quad P_n > S_n + 1 - n$

$P(2):$ from a) $\quad a_1 a_2 > a_1 + a_2 + 1 - 2 > 0$

i.e. $P_2 > S_2 + 1 - 2 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 2$, then $P_k > S_k + 1 - k$

$$P(k+1): \quad P_{k+1} - S_{k+1} - 1 + (k+1)$$
$$= a_{k+1} P_k - a_{k+1} - S_k - 1 + k + 1$$
$$= a_{k+1}(S_k + 1 - k) - a_{k+1} - S_k + k$$
$$= a_{k+1} S_k + \cancel{a_{k+1}} - a_{k+1} k - \cancel{a_{k+1}} - S_k + k$$
$$= (a_{k+1} - 1)(S_k - k)$$
$$= (a_{k+1} - 1) \left(\sum_{r=1}^{k} a_r - \sum_{r=1}^{k} 1 \right)$$
$$= (a_{k+1} - 1) \sum_{r=1}^{n} (a_r - 1) > 0 , \quad \therefore P_{k+1} > S_{k+1} + 1 - (k+1)$$

$\therefore$ By the PMI, $P(n)$ is true for all integers $n \geq 2$.