Mathematical Induction – Divisibility

Example $P(n): \quad 8 \mid 7^{2n-1} + 3^{2n}$ (which means $7^{2n-1} + 3^{2n}$ is divisible by 8)

$P(1): \quad 7^{2\cdot 1-1} + 3^{2\cdot 1} = 16 = 8 \cdot 2 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $7^{2k-1} + 3^{2k} = 8P$, for some $P \in \mathbb{Z}$

$P(k+1): \quad 7^{2(k+1)-1} + 3^{2(k+1)} = 49 \cdot 7^{2k-1} + 9 \cdot 3^{2k} = 40 \cdot 7^{2k-1} + 9 \cdot (7^{2k-1} + 3^{2k})$

$= 40 \cdot 7^{2k-1} + 9 \cdot 8P = 8(5 \cdot 7^{2k-1} + 9P) = 8Q$, where $Q = 5 \cdot 7^{2k-1} + 9P$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad 2 \mid n(n+1)$

$P(1): \quad 1 \cdot (1+1) = 2 \cdot 1 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $k(k+1) = 2P$, for some $P \in \mathbb{Z}$

$P(k+1): \quad (k+1)[(k+1)+1] = k(k+1) + 2(k+1) = 2P + 2(k+1) = 2Q$, where $Q = P + (k+1)$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad 3 \mid n(n+1)(n+2)$

$P(1): \quad 1 \cdot (1+1) \cdot (1+2) = 3 \cdot 2 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $k(k+1)(k+2) = 3P$, for some $P \in \mathbb{Z}$

$P(k+1): \quad (k+1)[(k+1)+1][(k+1)+2] = (k+1)(k+2)(k+3) = k(k+1)(k+2) + 3(k+1)(k+2)$

$= 3P + 3(k+1)(k+2) = 3Q$, where $Q = P + (k+1)(k+2)$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad 6 \mid n(n+1)(n+2)$

$P(1): \quad 1 \cdot (1+1) \cdot (1+2) = 6 \cdot 1 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $k(k+1)(k+2) = 6P$, for some $P \in \mathbb{Z}$

$P(k+1): \quad (k+1)[(k+1)+1][(k+1)+2] = (k+1)(k+2)(k+3) = k(k+1)(k+2) + 3(k+1)(k+2)$

$= 6P + 3(k+1)(k+2) = 6P + 3 \cdot 2Q$ (From previous example, $2 \mid (k+1)(k+2)$)

$= 6R$, where $R = P + Q$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad 9 \mid (3n+1) \cdot 7^n - 1$

$P(1): \quad (3 \cdot 1 + 1) \cdot 7^1 - 1 = 27 = 9 \cdot 3 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $(3k+1) \cdot 7^k - 1 = 9P$, for some $P \in \mathbb{Z}$

i.e. $-1 = 9P - (3k+1) \cdot 7^k$ The technique is to group all "non-k" terms on LHS.

$P(k+1): \quad [3(k+1)+1] \cdot 7^{k+1} - 1 = 7 \cdot (3k+4) \cdot 7^k + 9P - (3k+1) \cdot 7^k$

$= [(21k+28) - (3k+1)] \cdot 7^k + 9P = (18k+27) \cdot 7^k + 9P$

$= 9 \cdot (2k+3) \cdot 7^k + 9P = 9Q$, where $Q = (2k+3) \cdot 7^k + P$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad x^2 \mid (1+x)^n - nx - 1$

$P(1): \quad (1+x)^1 - 1 \cdot x - 1 = 0 = x^2 \cdot 0 , \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $(1+x)^k - kx - 1 = Px^2$, for some $P \in \mathbb{Z}$

$P(k+1): \quad (1+x)^{k+1} - (k+1)x - 1$

$= \left[(1+x)^k(1+x) - kx(x+1) - (x+1)\right] + [kx(x+1) + (x+1) - (k+1)x - 1]$

$= \left[(1+x)^k - kx - 1\right](x+1) + \left[kx^2 + kx + x + 1 - kx - x - 1\right]$

$= Px^2(x+1) + kx^2 = Qx^2$, where $Q = P(x+1) + k$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.