Mathematical Induction – Differentiation

Example $P(n): \quad \frac{d}{dx}x^n = nx^{n-1}$

$P(1): \quad LHS = \frac{d}{dx}x^1 = 1 \,, \quad RHS = 1 \cdot x^{1-1} = 1 \,, \quad LHS = RHS \,, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $\frac{d}{dx}x^k = kx^{k-1}$

$$P(k+1): \quad \frac{d}{dx}x^{k+1} = \frac{d}{dx}\left(x \cdot x^k\right) = x\frac{d}{dx}x^k + x^k\frac{d}{dx}x = x \cdot kx^{k-1} + x^k \cdot 1$$

$$= kx^k + x^k = (k+1)x^{(k+1)-1}$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad \frac{d^n}{dx^n}xe^x = (x+n)e^x$

$P(1): \quad LHS = \frac{d}{dx}xe^x = x\frac{d}{dx}e^x + e^x\frac{d}{dx}x = xe^x + e^x = (x+1)e^x = RHS \,, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $\frac{d^k}{dx^k}xe^x = (x+k)e^x$

$$P(k+1): \quad \frac{d^{k+1}}{dx^{k+1}}xe^x = \frac{d}{dx}\left(\frac{d^k}{dx^k}xe^x\right) = \frac{d}{dx}(x+k)e^x = (x+k)\frac{d}{dx}e^x + e^x\frac{d}{dx}(x+k)$$

$$=(x+k)e^x + e^x = [x+(k+1)]e^x$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad \frac{d^n}{dx^n}(e^x \sin x) = \left(\sqrt{2}\right)^n \sin\left(x + \frac{n\pi}{4}\right)e^x$

$$P(1): \quad LHS = \frac{d}{dx}(e^x \sin x) = e^x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}e^x = e^x \cos x + e^x \sin x$$

$$= e^x(\cos x + \sin x) = e^x \cdot \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right) = \frac{e^x}{\sqrt{2}}\left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right)$$

$$=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)e^x = RHS \,, \quad \therefore P(1) \text{ is true.}$$

If $P(k)$ is true up to some integer $k \geq 1$, then $\frac{d^k}{dx^k}(e^x \sin x) = \left(\sqrt{2}\right)^k \sin\left(x + \frac{k\pi}{4}\right)e^x$

$$P(k+1): \quad \frac{d^{k+1}}{dx^{k+1}}(e^x \sin x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(e^x \sin x)\right) = \frac{d}{dx}\left(\left(\sqrt{2}\right)^k \sin\left(x + \frac{k\pi}{4}\right)e^x\right)$$

$$= \left(\sqrt{2}\right)^k \left[\sin\left(x+\frac{k\pi}{4}\right)\frac{d}{dx}e^x + e^x\frac{d}{dx}\sin\left(x+\frac{k\pi}{4}\right)\right]$$

$$= \left(\sqrt{2}\right)^k \left[\sin\left(x+\frac{k\pi}{4}\right) + \cos\left(x+\frac{k\pi}{4}\right)\right]e^x$$

$$= \left(\sqrt{2}\right)^k \sqrt{2}\left[\sin\left(x+\frac{k\pi}{4}\right)\frac{1}{\sqrt{2}} + \cos\left(x+\frac{k\pi}{4}\right)\frac{1}{\sqrt{2}}\right]e^x$$

$$= \left(\sqrt{2}\right)^k \sqrt{2}\left[\sin\left(x+\frac{k\pi}{4}\right)\cos\frac{\pi}{4} + \cos\left(x+\frac{k\pi}{4}\right)\sin\frac{\pi}{4}\right]e^x$$

$$= \left(\sqrt{2}\right)^{k+1} \sin\left(x+\frac{(k+1)\pi}{4}\right)e^x$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.