Mathematical Induction

PMI: Principle of Mathematical Induction

$P(1)$ is true and if $P(k)$ is true then $P(k+1)$ is true. $\Rightarrow P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad \sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$P(1): \quad LHS = 1\,, \quad RHS = \frac{1}{2}\cdot 1 \cdot (1+1) = 1\,, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$.

$$P(k+1): \quad \sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = \frac{1}{2}k(k+1) + (k+1) = (k+1)\left[\frac{1}{2}k+1\right] = \frac{1}{2}(k+1)\left[(k+1)+1\right]$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad \sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$

$P(1): \quad LHS = 1^3 = 1\,, \quad RHS = 1^2 = 1\,, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $\sum_{r=1}^{k} r^3 = \left(\sum_{r=1}^{k} r\right)^2$.

$$P(k+1): \quad \sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3 = \left(\sum_{r=1}^{k} r\right)^2 + (k+1)^3$$

$$= \left[\left(\sum_{r=1}^{k} r\right)^2 + 2\left(\sum_{r=1}^{k} r\right)(k+1) + (k+1)^2\right] - 2\left(\sum_{r=1}^{k} r\right)(k+1) - (k+1)^2 + (k+1)^3$$

$$= \left[\left(\sum_{r=1}^{k} r\right) + (k+1)\right]^2 - \cancel{2}\left(\frac{1}{\cancel{2}}k(k+1)\right)(k+1) - (k+1)^2 + (k+1)^3$$

$$= \left(\sum_{r=1}^{k+1} r\right)^2 - (k+1)(k+1)^2 + (k+1)^3 = \left(\sum_{r=1}^{k+1} r\right)^2$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad \sum_{r=1}^{n} (-1)^{r+1} r = \frac{1}{4}\left[1-(-1)^n(2n+1)\right] \qquad \left(\text{LHS}=1-2+3-4+\ldots+(-1)^{n+1}n\right)$

$P(1): \quad LHS = 1\,, \quad RHS = \frac{1}{4}\left[1-(-1)^1(2\cdot 1+1)\right] = 1\,, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $\sum_{r=1}^{k} (-1)^{r+1} r = \frac{1}{4}\left[1-(-1)^k(2k+1)\right]$.

$$P(k+1): \quad \sum_{r=1}^{k+1} (-1)^{r+1} r = \sum_{r=1}^{k} (-1)^{r+1} r + (-1)^{(k+1)+1}(k+1)$$

$$= \frac{1}{4}\left[1-(-1)^k(2k+1)\right] + (-1)^k(k+1) = \frac{1}{4}\left[1-(-1)^k(2k+1) + 4(-1)^k(k+1)\right]$$

$$= \frac{1}{4}\left[1-(-1)^k\Big((2k+1)-4(k+1)\Big)\right] = \frac{1}{4}\left[1-(-1)^{(k+1)}\Big(2(k+1)+1\Big)\right]$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1} \qquad \left(\text{LHS}=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}\right)$

$P(1): \quad LHS = \frac{1}{1\cdot 3}, \quad RHS = \frac{1}{2\cdot 1+1} = \frac{1}{3}, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$.

$$P(k+1): \quad \sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} + \frac{1}{\Big(2(k+1)-1\Big)\Big(2(k+1)+1\Big)}$$

$$=\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{(2k^2+3k)+1}{(2k+1)(2k+3)} = \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)} = \frac{(k+1)}{2(k+1)+1}$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.

Example $P(n): \quad \frac{1}{x-1} - \sum_{r=1}^{n} \frac{1}{x^r} = \frac{1}{x^n(x-1)} \qquad \left(\text{LHS}=\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\ldots-\frac{1}{x^n}\right)$

$P(1): \quad LHS = \frac{1}{x-1} - \frac{1}{x} = \frac{1}{x(x-1)} = RHS, \quad \therefore P(1)$ is true.

If $P(k)$ is true up to some integer $k \geq 1$, then $\frac{1}{x-1} - \sum_{r=1}^{k} \frac{1}{x^r} = \frac{1}{x^k(x-1)}$.

$$P(k+1): \quad \frac{1}{x-1} - \sum_{r=1}^{k+1} \frac{1}{x^r} = \frac{1}{x-1} - \sum_{r=1}^{k} \frac{1}{x^r} - \frac{1}{x^{k+1}} = \frac{1}{x^k(x-1)} - \frac{1}{x^{k+1}} = \frac{x-(x-1)}{x^{k+1}(x-1)} = \frac{1}{x^{(k+1)}(x-1)}$$

$\therefore$ By the PMI, $P(n)$ is true for $n \in +\mathbb{Z}$.