Vectors

Basics: In this context, an n-dimensional vector is represented as $\mathbf{A} = (a_1, a_2, \ldots, a_n)$.

$\mathbf{U} = \mathbf{V} \quad \Leftrightarrow \quad u_r = v_r \quad \forall\, r \in \{\mathbb{N} : r \leq n\}$. $\quad \mathbf{0} = (0, 0, \ldots, 0)$. $\quad -\mathbf{U} = (-u_1, -u_2, \ldots, -u_n)$.

$\mathbf{U} + \mathbf{V} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$. $\quad \mathbf{U} - \mathbf{V} = \mathbf{U} + (-\mathbf{V})$. $\quad k\mathbf{U} = (ku_1, ku_2, \ldots, ku_n)$.

$\mathbf{U} + (\mathbf{V} + \mathbf{W}) = (\mathbf{U} + \mathbf{V}) + \mathbf{W}$. $\quad \mathbf{U} + \mathbf{V} = \mathbf{V} + \mathbf{U}$. $\quad \mathbf{U} + \mathbf{0} = \mathbf{0} + \mathbf{U} = \mathbf{U}$. $\quad \mathbf{U} + (-\mathbf{U}) = (-\mathbf{U}) + \mathbf{U} = \mathbf{0}$.

$c(\mathbf{U} + \mathbf{V}) = c\mathbf{U} + c\mathbf{V}$. $\quad (c + d)\mathbf{U} = c\mathbf{U} + d\mathbf{U}$. $\quad (cd)\mathbf{U} = c(d\mathbf{U})$. $\quad 1\mathbf{U} = \mathbf{U}$.

Presentation: In 3-dimension, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$.

$(a, b, c) = (a, 0, 0) + (0, b, 0) + (0, 0, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Length (magnitude) : $|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = \sqrt{a^2 + b^2 + c^2}$.

$|c\mathbf{U}| = |c||\mathbf{U}|$. $\quad |\mathbf{U} + \mathbf{V}| \leq |\mathbf{U}| + |\mathbf{V}|$ (The Triangle Inequality).

Dot product: $\mathbf{U} \cdot \mathbf{V} = \sum_{r=1}^{n} u_r v_r$.

$\mathbf{U} \cdot \mathbf{U} = |\mathbf{U}|^2$. $\quad$ Proof: $\mathbf{U} \cdot \mathbf{U} = u_1^2 + u_2^2 + u_3^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = |\mathbf{U}|^2$.

$\because |\mathbf{U} - \mathbf{V}|^2 = |\mathbf{U}|^2 + |\mathbf{V}|^2 - 2|\mathbf{U}||\mathbf{V}| \cos\theta$, where θ is the angle between $\mathbf{U}$ and $\mathbf{V}$,

$$\sum_{r=1}^{n}(u_r - v_r)^2 = \sum_{r=1}^{n} u_r^2 - 2\sum_{r=1}^{n} u_r v_r + \sum_{r=1}^{n} v_r^2 = \sum_{r=1}^{n} u_r^2 + \sum_{r=1}^{n} v_r^2 - 2|\mathbf{U}||\mathbf{V}| \cos\theta, \quad -2\sum_{r=1}^{n} u_r v_r = -2|\mathbf{U}||\mathbf{V}| \cos\theta,$$

$\therefore \mathbf{U} \cdot \mathbf{V} = |\mathbf{U}||\mathbf{V}| \cos\theta$. $\quad \theta = \cos^{-1}\left(\dfrac{\mathbf{U} \cdot \mathbf{V}}{|\mathbf{U}||\mathbf{V}|}\right)$. $\quad |\mathbf{U} \cdot \mathbf{V}| \leq |\mathbf{U}||\mathbf{V}|$ (The Cauchy-Schwarz Inequality).

Orthogonal: $\theta = \frac{\pi}{2} \Leftrightarrow \mathbf{U} \cdot \mathbf{V} = 0$. $\quad$ ($\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other.)

Parallel: $\theta = 0 \Leftrightarrow |\mathbf{U} \cdot \mathbf{V}| = |\mathbf{U}||\mathbf{V}|$.

Projection of $\mathbf{U}$ in the direction of $\mathbf{V}$: $\text{proj}_{\mathbf{v}}\mathbf{U} = \dfrac{\mathbf{U} \cdot \mathbf{V}}{\mathbf{V} \cdot \mathbf{V}}\mathbf{V}$. $\quad \mathbf{U} - \text{proj}_{\mathbf{v}}\mathbf{U}$ is orthogonal to $\mathbf{V}$.

For a unit vector $\hat{\mathbf{v}}$, $\text{proj}_{\hat{\mathbf{v}}}\mathbf{U} = (\mathbf{U} \cdot \mathbf{V})\mathbf{V}$.

Consider $\mathbf{U} \cdot \mathbf{i}$, the Direction Cosine of the x-axis is $\cos\alpha = \dfrac{\mathbf{U} \cdot \mathbf{i}}{|\mathbf{U}||\mathbf{i}|} = \dfrac{a}{|\mathbf{U}|}$.

Likewise, $\cos\beta = \dfrac{\mathbf{U} \cdot \mathbf{j}}{|\mathbf{U}||\mathbf{j}|} = \dfrac{b}{|\mathbf{U}|}$ for the y-axis, and $\cos\gamma = \dfrac{\mathbf{U} \cdot \mathbf{k}}{|\mathbf{U}||\mathbf{k}|} = \dfrac{c}{|\mathbf{U}|}$ for the z-axis.

Cross product: $\mathbf{U} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ (expressed in 3-dimensional as an example.)

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

$\mathbf{U} \times \mathbf{V}$ is orthogonal to both $\mathbf{U}$ and $\mathbf{V}$.

Proof: $(\mathbf{U} \times \mathbf{V}) \cdot \mathbf{U} = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = 0$. Likewise, $(\mathbf{U} \times \mathbf{V}) \cdot \mathbf{V} = 0$.

Note: $\mathbf{i} \times \mathbf{j} = (0 - 0, 0 - 0, 1 - 0) = \mathbf{k}$. $\mathbf{U} \times \mathbf{V}$ follows the xyz axis orientation, forming a right-hand system.

$\mathbf{U} \times (\mathbf{V} + \mathbf{W}) = \mathbf{U} \times \mathbf{V} + \mathbf{U} \times \mathbf{W}$. $\quad (\mathbf{U} + \mathbf{V}) \times \mathbf{W} = \mathbf{U} \times \mathbf{W} + \mathbf{V} \times \mathbf{W}$.

$(c\mathbf{U}) \times \mathbf{V} = c\mathbf{U} \times \mathbf{V} = \mathbf{U} \times (c\mathbf{V})$. $\quad \mathbf{0} \times \mathbf{U} = \mathbf{U} \times \mathbf{0} = \mathbf{0}$.

Watch-outs: $\mathbf{U} \times \mathbf{U} = \mathbf{0}$. $\quad \mathbf{U} \times \mathbf{V} = -\mathbf{V} \times \mathbf{U}$. $\quad \mathbf{U} \times (\mathbf{V} \times \mathbf{W}) \neq (\mathbf{U} \times \mathbf{V}) \times \mathbf{W}$. (So $\mathbf{U} \times \mathbf{V} \times \mathbf{W}$ is undefined.)

Theorem: $|\mathbf{U}\times\mathbf{V}|^{\mathbf{2}} = |\mathbf{U}|^{\mathbf{2}}|\mathbf{V}|^{\mathbf{2}} - (\mathbf{U}\cdot\mathbf{V})^{\mathbf{2}}$.

Proof: LHS $= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$

$= (u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2) + (u_3^2v_1^2 - 2u_3u_1v_3v_1 + u_1^2v_3^2) + (u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2)$

$= (u_2^2v_3^2 + u_3^2v_2^2 + u_3^2v_1^2 + u_1^2v_3^2 + u_1^2v_2^2 + u_2^2v_1^2) - 2(u_2u_3v_2v_3 + u_3u_1v_3v_1 + u_1u_2v_1v_2)$

RHS $= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$

$= (u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2) - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2)$

$- 2(u_1u_2v_1v_2 + u_2u_3v_2v_3 + u_3u_1v_3v_1) =$ LHS $\quad (QED)$

Theorem: $|\mathbf{U}\times\mathbf{V}| = |\mathbf{U}||\mathbf{V}|\sin\theta$, where θ is the angle between $\mathbf{U}$ and $\mathbf{V}$. ($\theta \in [0,\pi]$, so $\sin\theta \geq 0$.)

Proof: LHS $= \sqrt{|\mathbf{U}\times\mathbf{V}|^{\mathbf{2}}} = \sqrt{|\mathbf{U}|^{\mathbf{2}}|\mathbf{V}|^{\mathbf{2}} - (\mathbf{U}\cdot\mathbf{V})^{\mathbf{2}}} = \sqrt{|\mathbf{U}|^{\mathbf{2}}|\mathbf{V}|^{\mathbf{2}} - (|\mathbf{U}||\mathbf{V}|\cos\theta)^{\mathbf{2}}}$

$= \sqrt{|\mathbf{U}|^{\mathbf{2}}|\mathbf{V}|^{\mathbf{2}}(\mathbf{1} - \cos^{\mathbf{2}}\theta)} = |\mathbf{U}||\mathbf{V}|\sin\theta =$RHS.

For the parallelogram formed by $\mathbf{U}$ and $\mathbf{V}$, its height on $\mathbf{U}$ is $|\mathbf{V}|\sin\theta$, so its area is $|\mathbf{U}|\cdot|\mathbf{V}|\sin\theta$.

The area of the parallelogram formed by $\mathbf{U}$ and $\mathbf{V}$ is $|\mathbf{U}\times\mathbf{V}|$.

Triple Product: The volume of the parallelepiped formed by $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ is $(\mathbf{U}\times\mathbf{V})\cdot\mathbf{W} = \mathbf{U}\cdot(\mathbf{V}\times\mathbf{W})$.

$\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ are in right-hand system ($\mathbf{U}\times\mathbf{V}$ and $\mathbf{W}$ form an acute angle) to make a positive product.

Proof: Let $\mathbf{A} = \mathbf{U}\times\mathbf{V}$, then the area of the parallelogram formed by $\mathbf{U}$ and $\mathbf{V}$ is $|\mathbf{A}|$.

The height of the parallelepiped, H, is the projection of $\mathbf{W}$ on $\mathbf{A}$. $H = \dfrac{\mathbf{A}\cdot\mathbf{W}}{|\mathbf{A}|}$.

The volume of the parallelepiped is therefore $V = |\mathbf{A}|H = |\mathbf{A}|\dfrac{\mathbf{A}\cdot\mathbf{W}}{|\mathbf{A}|} = \mathbf{A}\cdot\mathbf{W} = (\mathbf{U}\times\mathbf{V})\cdot\mathbf{W}$.

Since $\mathbf{V}, \mathbf{W}$ and $\mathbf{U}$ are in the same right-hand system, $V = (\mathbf{V}\times\mathbf{W})\cdot\mathbf{U} = \mathbf{U}\cdot(\mathbf{V}\times\mathbf{W})$.

$\therefore \quad V = (\mathbf{U}\times\mathbf{V})\cdot\mathbf{W} = \mathbf{U}\cdot(\mathbf{V}\times\mathbf{W})$.