
$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

Solution: Let $\theta = \sin^{-1} x$, then $x = \sin\theta$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $x \in [-1, 1]$.

For $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos\theta \geq 0$, $\therefore \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$.

$$\int \sin^{-1} x \, dx = \int \theta \, dx = \theta x - \int x \, d\theta = x \sin^{-1} x - \int \sin\theta \, d\theta = x \sin^{-1} x + \cos\theta + C$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C \, .$$

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \sqrt{1-x^2} + C$$

Solution: Let $\theta = \cos^{-1} x$, then $x = \cos\theta$, where $\theta \in [0, \pi]$ and $x \in [-1, 1]$.

For $\theta \in [0, \pi]$, $\sin\theta \geq 0$, $\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}$.

$$\int \cos^{-1} x \, dx = \int \theta \, dx = \theta x - \int x \, d\theta = x \cos^{-1} x - \int \cos\theta \, d\theta = x \cos^{-1} x - \sin\theta + C$$

$$= x \cos^{-1} x - \sqrt{1-x^2} + C \, .$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \tfrac{1}{2} \ln\left(1+x^2\right) + C$$

Solution: Let $\theta = \tan^{-1} x$, then $x = \tan\theta$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $x \in \mathbb{R}$.

For $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos\theta \geq 0$, $\therefore \cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}} = \dfrac{1}{\sqrt{1+x^2}}$.

$$\int \tan^{-1} x \, dx = \int \theta \, dx = \theta x - \int x \, d\theta = x \tan^{-1} x - \int \tan\theta \, d\theta = x \tan^{-1} x - \int \frac{\sin\theta}{\cos\theta} \, d\theta$$

$$= x \tan^{-1} x + \ln(\cos\theta) + C = x \tan^{-1} x + \ln\left(\frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= x \tan^{-1} x - \tfrac{1}{2} \ln(1+x^2) + C$$

$$\int \cot^{-1} x \, dx = x \cot^{-1} x + \tfrac{1}{2} \ln\left(1+x^2\right) + C$$

Solution: Let $\theta = \cot^{-1} x$, then $x = \cot\theta$, where $\theta \in [0, \pi]$ and $x \in \mathbb{R}$.

For $\theta \in [0, \pi]$, $\sin\theta \geq 0$, $\therefore \sin\theta = \dfrac{1}{\sqrt{1+\cot^2\theta}} = \dfrac{1}{\sqrt{1+x^2}}$.

$$\int \cot^{-1} x \, dx = \int \theta \, dx = \theta x - \int x \, d\theta = x \cot^{-1} x - \int \cot\theta \, d\theta = x \cot^{-1} x - \int \frac{\cos\theta}{\sin\theta} \, d\theta$$

$$= x \cot^{-1} x - \ln(\sin\theta) + C = x \cot^{-1} x - \ln\left(\frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= x \cot^{-1} x + \tfrac{1}{2} \ln(1+x^2) + C$$

$$\int \sec^{-1} x\, dx = x\sec^{-1} x - \frac{x}{|x|} \ln\left|x+\sqrt{x^2-1}\right| + C$$

Solution: Let $\theta = \sec^{-1} x$, then $x = \sec\theta$, where $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and $|x| \geq 1$.

Let $I = \displaystyle\int \sec^{-1} x\, dx = \int \theta\, dx = \theta x - \int x\, d\theta = x\sec^{-1} x - \int \sec\theta\, d\theta$

$= x\sec^{-1} x - \displaystyle\int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta+\tan\theta}\, d\theta$

$= x\sec^{-1} x - \displaystyle\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$

$= x\sec^{-1} x - \displaystyle\int \frac{(\sec\theta+\tan\theta)'}{\sec\theta+\tan\theta}\, d\theta$

$= x\sec^{-1} x - \ln|\sec\theta+\tan\theta| + C$

For $\theta \in [0, \frac{\pi}{2})$, $x = \sec\theta \geq 1$ and $\tan\theta \geq 0$, $\therefore\ |\sec\theta+\tan\theta| = \left|x+\sqrt{x^2-1}\right|$.

$I = x\sec^{-1} x - \ln|\sec\theta+\tan\theta| + C = x\sec^{-1} x - \ln\left|x+\sqrt{x^2-1}\right| + C \quad \ldots(1)$

For $\theta \in (\frac{\pi}{2}, \pi]$, $x = \sec\theta \leq -1$ and $\tan\theta \leq 0$, $\therefore\ |\sec\theta+\tan\theta| = \left|x-\sqrt{x^2-1}\right|$

$= \left|\left(x-\sqrt{x^2-1}\right)\cdot\dfrac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right| = \left|\dfrac{x^2-(x^2-1)}{x+\sqrt{x^2-1}}\right| = \dfrac{1}{|x+\sqrt{x^2-1}|}$

$I = x\sec^{-1} x - \ln|\sec\theta+\tan\theta| + C = x\sec^{-1} x - \ln\dfrac{1}{|x+\sqrt{x^2-1}|} + C$

$= x\sec^{-1} x + \ln\left|x+\sqrt{x^2-1}\right| + C \quad \ldots(2)$

From (1) and (2): $\displaystyle\int \sec^{-1} x\, dx = x\sec^{-1} x - \frac{x}{|x|}\ln\left|x+\sqrt{x^2-1}\right| + C$

$$\int \csc^{-1} x\, dx = x\csc^{-1} x + \frac{x}{|x|} \ln\left|x+\sqrt{x^2-1}\right| + C$$

Solution: Let $\theta = \csc^{-1} x$, then $x = \csc\theta$, where $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ and $|x| \geq 1$.

Let $I = \displaystyle\int \csc^{-1} x\, dx = \int \theta\, dx = \theta x - \int x\, d\theta = x\csc^{-1} x - \int \csc\theta\, d\theta$

$= x\csc^{-1} x - \displaystyle\int \csc\theta \cdot \frac{\csc\theta - \cot\theta}{\csc\theta-\cot\theta}\, d\theta$

$= x\csc^{-1} x - \displaystyle\int \frac{\csc^2\theta - \csc\theta\cot\theta}{\csc\theta-\cot\theta}\, d\theta$

$= x\csc^{-1} x - \displaystyle\int \frac{(\csc\theta-\cot\theta)'}{\csc\theta-\cot\theta}\, d\theta$

$= x\csc^{-1} x - \ln|\csc\theta-\cot\theta| + C$

For $\theta \in [-\frac{\pi}{2}, 0)$, $x \leq -1$, $\cot\theta \leq 0$, $\cot\theta = -\sqrt{\csc^2\theta - 1} = -\sqrt{x^2-1}$, $\therefore\ |\csc\theta-\cot\theta| = \left|x+\sqrt{x^2-1}\right|$.

$I = x\csc^{-1} x - \ln|\csc\theta-\cot\theta| + C = x\csc^{-1} x - \ln\left|x+\sqrt{x^2-1}\right| + C \quad \ldots(1)$

For $\theta \in (0, \frac{\pi}{2}]$, $x \geq 1$, $\cot\theta \geq 0$, $\cot\theta = \sqrt{\csc^2\theta - 1} = \sqrt{x^2-1}$, $\therefore\ |\csc\theta-\cot\theta| = \left|x-\sqrt{x^2-1}\right|$

$= \left|\left(x-\sqrt{x^2-1}\right)\cdot\dfrac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right| = \left|\dfrac{x^2-(x^2-1)}{x+\sqrt{x^2-1}}\right| = \dfrac{1}{|x+\sqrt{x^2-1}|}$

$I = x\csc^{-1} x - \ln|\csc\theta-\cot\theta| + C = x\csc^{-1} x - \ln\dfrac{1}{|x+\sqrt{x^2-1}|} + C$

$= x\csc^{-1} x + \ln\left|x+\sqrt{x^2-1}\right| + C \quad \ldots(2)$

From (1) and (2): $\displaystyle\int \csc^{-1} x\, dx = x\csc^{-1} x + \frac{x}{|x|}\ln\left|x+\sqrt{x^2-1}\right| + C$