Integrals of sin and cos special forms

$$\int \frac{dx}{a\cos x + b\sin x + c} = -\frac{2}{(c-a)\tan\frac{x}{2} + b} + C\,, \quad \text{where } c^2 = a^2 + b^2$$

$$\int \frac{dx}{a\cos x + b\sin x + c} = \frac{2}{\sqrt{c^2 - a^2 - b^2}} \tan^{-1}\left[\frac{(c-a)\tan\frac{x}{2} + b}{\sqrt{c^2 - a^2 - b^2}}\right] + C\,, \quad \text{where } c^2 > a^2 + b^2$$

$$\int \frac{dx}{a\cos x + b\sin x + c} = \frac{1}{\sqrt{a^2 + b^2 - c^2}} \ln\left|\frac{(c-a)\tan\frac{x}{2} + b - \sqrt{a^2 + b^2 - c^2}}{(c-a)\tan\frac{x}{2} + b + \sqrt{a^2 + b^2 - c^2}}\right| + C\,, \quad \text{where } c^2 < a^2 + b^2$$

$$\text{Let} \quad t = \tan\frac{x}{2}\,, \quad u = (c-a)\,t + b\,, \quad du = (c-a)\,dt\,, \quad k = \sqrt{|c^2 - a^2 - b^2|}$$

$$\begin{aligned}
I &= \int \frac{\frac{2}{1+t^2}\,dt}{a\frac{1-t^2}{1+t^2} + b\frac{2t}{1+t^2} + c} \\
&= \int \frac{2\,dt}{a\,(1-t^2) + 2bt + c\,(1+t^2)} \\
&= \int \frac{2\,dt}{a - at^2 + 2bt + c + ct^2} \\
&= \int \frac{2\,dt}{(c-a)\,t^2 + 2bt + c + a} \\
&= \int \frac{2\,(c-a)\ dt}{(c-a)^2\,t^2 + 2b\,(c-a)\,t + c^2 - a^2} \\
&= \int \frac{2\,(c-a)\ dt}{\left[(c-a)^2\,t^2 + 2b\,(c-a)\,t + b^2\right] - b^2 + c^2 - a^2} \\
&= \int \frac{2\,(c-a)\ dt}{[(c-a)\,t + b]^2 + [c^2 - a^2 - b^2]}
\end{aligned}$$

Case 1: $c^2 = a^2 + b^2\,, \quad k = \sqrt{c^2 - a^2 - b^2}\,, \quad k^2 = c^2 - a^2 - b^2$

$$\begin{aligned}
I &= \int \frac{2\,du}{u^2} \\
&= -\frac{2}{u} + C \\
&= -\frac{2}{(c-a)\tan\frac{x}{2} + b} + C
\end{aligned}$$

Case 2: $c^2 > a^2 + b^2\,, \quad k = \sqrt{c^2 - a^2 - b^2}\,, \quad k^2 = c^2 - a^2 - b^2$

$$\begin{aligned}
I &= \int \frac{2\,du}{u^2 + k^2} \\
&= \frac{2}{k} \tan^{-1}\frac{u}{k} + C \\
&= \frac{2}{\sqrt{c^2 - a^2 - b^2}} \tan^{-1}\left[\frac{(c-a)\tan\frac{x}{2} + b}{\sqrt{c^2 - a^2 - b^2}}\right] + C
\end{aligned}$$

Case 3: $c^2 < a^2 + b^2\,, \quad k = \sqrt{a^2 + b^2 - c^2}\,, \quad k^2 = a^2 + b^2 - c^2$

$$\begin{aligned}
I &= \int \frac{2\,du}{u^2 - k^2} \\
&= \frac{1}{k} \int \frac{1}{u-k} - \frac{1}{u+k}\,dx \\
&= \frac{1}{k} \ln\left|\frac{u-k}{u+k}\right| + C \\
&= \frac{1}{\sqrt{a^2 + b^2 - c^2}} \ln\left|\frac{(c-a)\tan\frac{x}{2} + b - \sqrt{a^2 + b^2 - c^2}}{(c-a)\tan\frac{x}{2} + b + \sqrt{a^2 + b^2 - c^2}}\right| + C
\end{aligned}$$