Table of Integrals – Trigonometric Functions – The Working Out

Note: $t = \tan\frac{ax}{2}$, $\sin ax = \frac{2t}{1+t^2}$, $\cos ax = \frac{1-t^2}{1+t^2}$, $\tan ax = \frac{2t}{1-t^2}$, $dx = \frac{1}{a}\cdot\frac{2}{1+t^2}\,dt$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \tan ax\,dx = \int \frac{\sin ax}{\cos ax}\,dx = \int \frac{-\frac{1}{a}(\cos ax)'}{\cos ax}\,dx = -\frac{1}{a}\ln|\cos ax| + C \quad \Big(= \frac{1}{a}\ln|\sec ax| + C\Big)$$

$$\int \cot ax\,dx = \int \frac{\cos ax}{\sin ax}\,dx = \int \frac{\frac{1}{a}(\sin ax)'}{\sin ax}\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \int \sec ax\cdot\frac{\sec ax + \tan ax}{\sec ax + \tan ax}\,dx = \int \frac{\sec^2 ax + \sec ax\tan ax}{\sec ax + \tan ax}\,dx$$
$$= \int \frac{\frac{1}{a}(\tan ax + \sec ax)'}{\sec ax + \tan ax}\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sec ax\,dx = \int \frac{\cancel{1+t^2}}{1-t^2}\cdot\frac{1}{a}\cdot\frac{2}{\cancel{1+t^2}}\,dt = \frac{1}{a}\int\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt$$
$$= \frac{1}{a}\Big(\ln|1+t| - \ln|1-t|\Big) + C = \frac{1}{a}\ln\left|\frac{1+t}{1-t}\right| + C = \frac{1}{a}\ln\left|\frac{1+\tan\frac{ax}{2}}{1-\tan\frac{ax}{2}}\right| + C$$

$$\int \csc ax\,dx = \int \csc ax\cdot\frac{\csc ax - \cot ax}{\csc ax - \cot ax}\,dx = \int \frac{\csc^2 ax - \csc ax\cot ax}{\csc ax - \cot ax}\,dx$$
$$= \int \frac{\frac{1}{a}(\cot ax - \csc ax)'}{\csc ax - \cot ax}\,dx = \frac{1}{a}\ln|\csc ax - \cot ax| + C \quad \Big(= -\frac{1}{a}\ln|\csc ax + \cot ax| + C\Big)$$

$$\int \csc ax\,dx = \int \frac{\cancel{1+t^2}}{\cancel{2}t}\cdot\frac{1}{a}\cdot\frac{\cancel{2}}{\cancel{1+t^2}}\,dt = \frac{1}{a}\int\frac{1}{t}\,dt = \frac{1}{a}\ln|t| + C = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right| + C$$

$$I_n = \int \sin^n ax\,dx \quad\Rightarrow\quad I_n = \tfrac{n-1}{n}I_{n-2} - \tfrac{1}{na}\sin^{n-1} ax\cdot\cos ax$$
$$I_n = \int \sin^{n-1} ax\cdot\sin ax\,dx$$
$$= \sin^{n-1} ax\cdot\left(-\frac{1}{a}\cos ax\right) - \int\left(-\frac{1}{\cancel{a}}\cos ax\right)\cdot(n-1)\sin^{n-2} ax\cdot\cos ax\cdot\cancel{a}\,dx$$
$$= -\frac{1}{a}\sin^{n-1} ax\cdot\cos ax + (n-1)\int\sin^{n-2} ax\cdot\cos^2 ax\,dx$$
$$= -\frac{1}{a}\sin^{n-1} ax\cdot\cos ax + (n-1)\int\sin^{n-2} ax\cdot(1-\sin^2 ax)\,dx$$
$$= -\frac{1}{a}\sin^{n-1} ax\cdot\cos ax + (n-1)\int\sin^{n-2} ax\,dx - (n-1)\int\sin^n ax\,dx$$
$$= -\frac{1}{a}\sin^{n-1} ax\cdot\cos ax + (n-1)I_{n-2} - (n-1)I_n$$
$$nI_n = -\frac{1}{a}\sin^{n-1} ax\cdot\cos ax + (n-1)I_{n-2}$$
$$I_n = \tfrac{n-1}{n}I_{n-2} - \tfrac{1}{na}\sin^{n-1} ax\cdot\cos ax$$

$$I_n = \int \cos^n ax\,dx \quad \Rightarrow \quad I_n = \tfrac{n-1}{n} I_{n-2} + \tfrac{1}{na}\cos^{n-1} ax \cdot \sin ax$$

$$\begin{aligned}
I_n &= \int \cos^{n-1} ax \cdot \cos ax\,dx \\
&= \cos^{n-1} ax \cdot \left(\frac{1}{a}\sin ax\right) - \int \left(\frac{1}{\cancel{a}}\sin ax\right) \cdot (n-1)\cos^{n-2} ax \cdot (-\sin ax) \cdot \cancel{a}\,dx \\
&= \frac{1}{a}\cos^{n-1} ax \cdot \sin ax + (n-1)\int \cos^{n-2} ax \cdot \sin^2 ax\,dx \\
&= \frac{1}{a}\cos^{n-1} ax \cdot \sin ax + (n-1)\int \cos^{n-2} ax \cdot (1-\cos^2 ax)\,dx \\
&= \frac{1}{a}\cos^{n-1} ax \cdot \sin ax + (n-1)\int \cos^{n-2} ax\,dx - (n-1)\int \cos^n ax\,dx \\
&= \frac{1}{a}\cos^{n-1} ax \cdot \sin ax + (n-1)I_{n-2} - (n-1)I_n \\
nI_n &= \frac{1}{a}\cos^{n-1} ax \cdot \sin ax + (n-1)I_{n-2} \\
I_n &= \tfrac{n-1}{n} I_{n-2} + \tfrac{1}{na}\cos^{n-1} ax \cdot \sin ax
\end{aligned}$$

$$I_n = \int \tan^n ax\,dx \quad \Rightarrow \quad I_n = \tfrac{1}{(n-1)a}\tan^{n-1} ax - I_{n-2}$$

$$\begin{aligned}
I_n &= \int \left(\frac{\tan^{n-1} ax}{\sec ax}\right) \cdot \sec ax\ \tan ax\,dx \\
&= \left(\frac{\tan^{n-1} ax}{\sec ax}\right) \cdot \frac{1}{a}\sec ax - \int \frac{1}{\cancel{a}}\sec ax \cdot \frac{1}{\sec^2 ax}\left(\sec ax \cdot (n-1)\tan^{n-2} ax\ \sec^2 ax \cdot \cancel{a} - \tan^{n-1} ax \cdot \sec ax\ \tan ax \cdot \cancel{a}\right)\,dx \\
&= \frac{1}{a}\tan^{n-1} ax - \int \left[(n-1)\tan^{n-2} ax\ \sec^2 ax - \tan^n ax\right]\,dx \\
&= \frac{1}{a}\tan^{n-1} ax - \int (n-1)\tan^{n-2} ax \cdot (\tan^2 ax + 1) + \int \tan^n ax\,dx \\
&= \frac{1}{a}\tan^{n-1} ax - \int (n-1)\tan^n ax\,dx - \int (n-1)\tan^{n-2} ax\,dx + \int \tan^n ax\,dx \\
&= \frac{1}{a}\tan^{n-1} ax - (n-1)I_n - (n-1)I_{n-2} + I_n \\
(n-1)I_n &= \frac{1}{a}\tan^{n-1} ax - (n-1)I_{n-2} \\
I_n &= \tfrac{1}{(n-1)a}\tan^{n-1} ax - I_{n-2}
\end{aligned}$$

$$I_n = \int \sec^n ax\,dx \quad \Rightarrow \quad I_n = \tfrac{1}{(n-1)a}\sec^{n-2} ax \cdot \tan ax + \tfrac{n-2}{n-1} I_{n-2}$$

$$\begin{aligned}
I_n &= \int \sec^{n-2} ax \cdot \sec^2 ax\,dx \\
&= \sec^{n-2} ax \cdot \frac{1}{a}\tan ax - \int \frac{1}{\cancel{a}}\tan ax \cdot (n-2)\sec^{n-3} ax \cdot \sec ax\ \tan ax \cdot \cancel{a}\,dx \\
&= \frac{1}{a}\sec^{n-2} ax\ \tan ax - (n-2)\int \sec^{n-2} ax \cdot \tan^2 ax\,dx \\
&= \frac{1}{a}\sec^{n-2} ax\ \tan ax - (n-2)\int \sec^{n-2} ax \cdot (\sec^2 ax - 1)\,dx \\
&= \frac{1}{a}\sec^{n-2} ax\ \tan ax - (n-2)\int \sec^n ax\,dx + (n-2)\int \sec^{n-2} ax\,dx \\
&= \frac{1}{a}\sec^{n-2} ax\ \tan ax - (n-2)I_n + (n-2)I_{n-2} \\
(n-1)I_n &= \frac{1}{a}\sec^{n-2} ax\ \tan ax + (n-2)I_{n-2} \\
I_n &= \tfrac{1}{(n-1)a}\sec^{n-2} ax \cdot \tan ax + \tfrac{n-2}{n-1} I_{n-2}
\end{aligned}$$