Table of Integrals – The Working Out

$$\int \frac{1}{a^2+b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\frac{bx}{a}\,, \quad a>0\,, \quad b>0$$

$$\int \frac{1}{a^2-b^2x^2}\,dx = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right|\,, \quad a>0\,, \quad b>0 \qquad \left(=\frac{1}{ab}\tanh^{-1}\frac{bx}{a}\,, \text{ when } -\frac{a}{b}<x<\frac{a}{b}\right)$$

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\frac{bx}{a}\,, \quad a>0\,, \quad b>0\,, \quad -\frac{a}{b}<x<\frac{a}{b}$$

$$\int \frac{1}{\sqrt{b^2x^2-a^2}}\,dx = \frac{1}{b}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|\,, \quad a>0\,, \quad b>0\,, \quad 0<\frac{a}{b}<|x|$$

$$\int \frac{1}{\sqrt{b^2x^2+a^2}}\,dx = \frac{1}{b}\ln\left|bx+\sqrt{b^2x^2+a^2}\right|\,, \quad a>0\,, \quad b>0$$

$$\int \sqrt{a^2-b^2x^2}\,dx = \frac{x}{2}\sqrt{a^2-b^2x^2}+\frac{a^2}{2b}\sin^{-1}\frac{bx}{a}\,, \quad a>0\,, \quad b>0\,, \quad -\frac{a}{b}\le x\le\frac{a}{b}$$

$$\int \sqrt{b^2x^2-a^2}\,dx = \frac{x}{2}\sqrt{b^2x^2-a^2}-\frac{a^2}{2b}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|\,, \quad a>0\,, \quad b>0\,, \quad 0<\frac{a}{b}\le|x|$$

$$\int \sqrt{b^2x^2+a^2}\,dx = \frac{x}{2}\sqrt{b^2x^2+a^2}+\frac{a^2}{2b}\ln\left|bx+\sqrt{b^2x^2+a^2}\right|\,, \quad a>0\,, \quad b>0$$

Find $\displaystyle\int \sqrt{a^2-b^2x^2}\,dx\,, \quad a>0\,, \quad b>0\,, \quad -\frac{a}{b}\le x\le\frac{a}{b}$

$-1\le\frac{b}{a}x\le 1\,, \quad$ Let $\theta=\sin^{-1}\frac{b}{a}x \quad \left(\text{where } -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)\,, \qquad \sin\theta=\frac{b}{a}x\,, \quad \cos\theta\,d\theta=\frac{b}{a}dx$

$$\sqrt{a^2-b^2x^2}=a\sqrt{1-\left(\frac{bx}{a}\right)^2}=a\sqrt{1-\sin^2\theta}=|a|\cos\theta\ge 0 \quad \text{for } -\tfrac{\pi}{2}\le\theta\le\tfrac{\pi}{2}\,.$$

$$\int \sqrt{a^2-b^2x^2}\,dx=\int a\cos\theta\cdot\frac{a}{b}\cos\theta\,d\theta=\frac{a^2}{b}\int\cos^2\theta\,d\theta$$

$$=\frac{a^2}{b}\int\frac{1+\cos 2\theta}{2}\,d\theta=\frac{a^2}{2b}\left(\theta+\frac{\sin 2\theta}{2}\right)+C=\frac{a^2}{2b}\left(\sin^{-1}\frac{b}{a}x+\sin\theta\cos\theta\right)+C$$

$$=\frac{a^2}{2b}\left(\sin^{-1}\frac{b}{a}x+\frac{b}{a}x\sqrt{1-\left(\frac{bx}{a}\right)^2}\right)+C=\frac{a^2}{2b}\sin^{-1}\frac{b}{a}x+\frac{x}{2}\sqrt{a^2-b^2x^2}+C$$

$$=\frac{x}{2}\sqrt{a^2-b^2x^2}+\frac{a^2}{2b}\sin^{-1}\frac{bx}{a}+C$$