
$$I = \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C, \quad x \in \mathbb{R}.$$

Solution: Since $1 + \tan^2\theta = \sec^2\theta$, let $x = \tan\theta$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $x \in \mathbb{R}$.

$dx = d(\tan\theta) = \sec^2\theta\, d\theta$ and $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

$$I = \int \frac{1}{\sec^2\theta} \cdot \sec^2\theta\, d\theta = \int d\,\theta = \theta + C = \tan^{-1} x + C.$$

$$I = \int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C, \quad x \neq \pm 1.$$

Solution: $I = \frac{1}{2}\int \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx = \frac{1}{2}\left(\int \frac{1}{1+x}\,dx + \int \frac{1}{1-x}\,dx\right)$

$$= \frac{1}{2}\Big(\ln|1+x| - \ln|1-x|\Big) + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

(Note: I is undefined when $x = \pm 1$. So any definite integrals must avoid having ± 1 in the interval of integration.)

$$I = \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C, \quad -1 < x < 1.$$

Solution: Since $1 - \sin^2\theta = \cos^2\theta$, let $x = \sin\theta$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $x \in (-1, 1)$.

$dx = d(\sin\theta) = \cos\theta\, d\theta$ and $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1-x^2} > 0$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$$I = \int \frac{1}{\cos\theta} \cdot \cos\theta\, d\,\theta = \int d\,\theta = \theta = \sin^{-1} x + C.$$

$$I = \int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x + \sqrt{x^2-1}\right| + C, \quad |x| > 1$$

Solution: Since $\sec^2\theta - 1 = \tan^2\theta$, let $x = \sec\theta$, where $\theta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$, $|x| > 1$.

$dx = d(\sec\theta) = \sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = |\tan\theta|$.

$I = \int \frac{1}{|\tan\theta|} \cdot \sec\theta\tan\theta\, d\theta = \int \frac{1}{+\tan\theta} \cdot \sec\theta\tan\theta\, d\theta = \int \sec\theta\, d\theta$ for $\theta \in (0, \frac{\pi}{2})$,

or $I = \int \frac{1}{-\tan\theta} \cdot \sec\theta\tan\theta\, d\theta = -\int \sec\theta\, d\theta$ for $\theta \in (\frac{\pi}{2}, \pi)$.

Given $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$,

For $\theta \in (0, \frac{\pi}{2})$, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{x^2-1}$, $I = \ln\left|x + \sqrt{x^2-1}\right| + C$.

For $\theta \in (\frac{\pi}{2}, \pi)$, $\tan\theta = -\sqrt{\sec^2\theta - 1} = -\sqrt{x^2-1}$, $I = -\ln\left|x - \sqrt{x^2-1}\right| + C$

$$= \ln\left|\frac{1}{x - \sqrt{x^2-1}}\right| + C = \ln\left|\frac{x + \sqrt{x^2-1}}{x^2 - (x^2-1)}\right| + C = \ln\left|x + \sqrt{x^2-1}\right| + C.$$

Same result in both cases. $\therefore I = \ln\left|x + \sqrt{x^2-1}\right| + C.$

(Note: $\because |x| > 1$, $\therefore \left(\sqrt{x^2-1}\right)^2 = \left|x^2-1\right| = x^2 - 1 > 0.$)

$$I=\int\frac{1}{\sqrt{x^2+1}}\,dx=\ln\left|x+\sqrt{x^2+1}\right|+C,\quad x\in\mathbb{R}.$$

Solution: Since $\tan^2\theta+1=\sec^2\theta,\quad$ let $x=\tan\theta,$ where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2}),\ x\in\mathbb{R}.$

$dx=d\,(\tan\theta)=\sec^2\theta\,d\theta\quad$ and $\quad\sec\theta=\sqrt{\tan^2\theta+1}=\sqrt{x^2+1}>0\quad$ for $\theta\in(-\frac{\pi}{2},\frac{\pi}{2}).$

$$I=\int\frac{1}{\sec\theta}\cdot\sec^2\theta\,d\theta=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C=\ln\left|x+\sqrt{x^2+1}\right|+C.$$

$$I=\int\sqrt{1-x^2}\,dx=\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+C,\quad -1\le x\le 1$$

Solution: Since $1-\sin^2\theta=\cos^2\theta,\quad$ let $x=\sin\theta,$ where $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}],\ x\in[-1,1].$

$dx=d\,(\sin\theta)=\cos\theta\,d\theta\quad$ and $\quad\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}>0\quad$ for $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}].$

$$I=\int\cos\theta\cdot\cos\theta\,d\,\theta=\int\cos^2\theta\,d\,\theta=\int\frac{\cos(2\theta)+1}{2}d\,\theta$$

$$=\frac{1}{4}\sin(2\theta)+\frac{\theta}{2}+C=\frac{1}{4}\cdot2\sin\theta\cos\theta+\frac{\theta}{2}+C=\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+C.$$

$$I=\int\sqrt{x^2-1}\,dx=\frac{x}{2}\sqrt{x^2-1}-\frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+C,\quad |x|\ge 1$$

Solution: Since $\sec^2\theta-1=\tan^2\theta,\quad$ let $x=\sec\theta,$ where $\theta\in(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi),\ |x|\ge 1.$

$dx=d\,(\sec\theta)=\sec\theta\tan\theta\,d\theta\quad$ and $\quad\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=|\tan\theta|.$

$$I=\int|\tan\theta|\cdot\sec\theta\tan\theta\,d\theta=\int+\tan\theta\cdot\sec\theta\tan\theta\,d\theta=\int\sec\theta\tan^2\theta\,d\theta\quad\text{for }\theta\in(0,\tfrac{\pi}{2}),$$

or $I=\int-\tan\theta\cdot\sec\theta\tan\theta\,d\theta=-\int\sec\theta\tan^2\theta\,d\theta\quad$ for $\theta\in(\frac{\pi}{2},\pi).$

Let $I_2=\int\sec\theta\tan^2\theta\,d\theta=\int\sec\theta(\sec^2\theta-1)\,d\theta=\int\sec^3\theta\,d\theta-\int\sec\theta\,d\theta=\left[\int\sec\theta\,d(\tan\theta)\right]-\int\sec\theta\,d\theta$

$$=\left[\sec\theta\tan\theta-\int\tan\theta\,d(\sec\theta)\right]-\int\sec\theta\,d\theta=\left[\sec\theta\tan\theta-\int\tan\theta\cdot\sec\theta\tan\theta\,d\theta\right]-\int\sec\theta\,d\theta$$

$$=\left[\sec\theta\tan\theta-\int\sec\theta\tan^2\theta\,d\theta\right]-\int\sec\theta\,d\theta=[\sec\theta\tan\theta-I_2]-\int\sec\theta\,d\theta=-I_2+\left(\sec\theta\tan\theta-\int\sec\theta\,d\theta\right).$$

$$\because 2\,I_2=\left(\sec\theta\tan\theta-\int\sec\theta\,d\theta\right),\qquad\therefore I_2=\frac{1}{2}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.$$

For $\theta\in(0,\frac{\pi}{2}),\ \tan\theta=\sqrt{\sec^2\theta-1}=\sqrt{x^2-1},\quad I=I_2=\frac{1}{2}\left(x\sqrt{x^2-1}-\ln\left|x+\sqrt{x^2-1}\right|\right)+C.$

For $\theta\in(\frac{\pi}{2},\pi),\ \tan\theta=-\sqrt{\sec^2\theta-1}=-\sqrt{x^2-1},\quad I=-I_2=-\frac{1}{2}\left(-x\sqrt{x^2-1}-\ln\left|x-\sqrt{x^2-1}\right|\right)+C$

$$=\frac{1}{2}\left(x\sqrt{x^2-1}+\ln\left|x-\sqrt{x^2-1}\right|\right)+C=\frac{1}{2}\left(x\sqrt{x^2-1}-\ln\left|\frac{1}{x-\sqrt{x^2-1}}\right|\right)+C$$

$$=\frac{1}{2}\left(x\sqrt{x^2-1}-\ln\left|\frac{x+\sqrt{x^2-1}}{x^2-(x^2-1)}\right|\right)+C=\frac{1}{2}\left(x\sqrt{x^2-1}-\ln\left|x+\sqrt{x^2-1}\right|\right)+C.$$

Same result in both cases. $\quad\therefore I=\frac{1}{2}\left(x\sqrt{x^2-1}-\ln\left|x+\sqrt{x^2-1}\right|\right)+C=\frac{x}{2}\sqrt{x^2-1}-\frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+C.$

(Note: $\because|x|>1,\quad\therefore\left(\sqrt{x^2-1}\right)^2=\left|x^2-1\right|=x^2-1>0.$)

$$I=\int\sqrt{x^2+1}\,dx=\frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C,\quad x\in\mathbb{R}.$$

Solution: Since $\tan^2\theta+1=\sec^2\theta$, let $x=\tan\theta$, where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, $x\in\mathbb{R}$.

$dx=d\,(\tan\theta)=\sec^2\theta\,d\theta$ and $\sec\theta=\sqrt{\tan^2\theta+1}=\sqrt{x^2+1}>0$ for $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

$$I=\int\sec\theta\cdot\sec^2\theta\,d\theta=\int\sec\theta\cdot(\tan^2\theta+1)\,d\theta$$

$$=\left[\int\sec\theta\tan^2\theta\,d\theta\right]+\int\sec\theta\,d\theta=\left[\frac{1}{2}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)\right]+\ln|\sec\theta+\tan\theta|+C$$

$$=\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|+C=\frac{1}{2}\tan\theta\sec\theta+\frac{1}{2}\ln|\tan\theta+\sec\theta|+C$$

$$=\frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\ln|x+\sqrt{x^2+1}|+C.$$

(Note: From previous solutions: $I_2=\int\sec\theta\tan^2\theta\,d\theta=\frac{1}{2}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.$)

Find $\int\sqrt{a^2-b^2x^2}\,dx$, $a>0$, $b>0$, $-\frac{a}{b}\le x\le\frac{a}{b}$

$-1\le\frac{b}{a}x\le1$, Let $\theta=\sin^{-1}\frac{b}{a}x$ $\left(\text{where }-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)$, $\sin\theta=\frac{b}{a}x$, $\cos\theta\,d\theta=\frac{b}{a}dx$

$\sqrt{a^2-b^2x^2}=a\sqrt{1-\left(\frac{bx}{a}\right)^2}=a\sqrt{1-\sin^2\theta}=|a|\cos\theta\ge0$ for $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

$$\int\sqrt{a^2-b^2x^2}\,dx=\int a\cos\theta\cdot\frac{a}{b}\cos\theta\,d\theta=\frac{a^2}{b}\int\cos^2\theta\,d\theta$$

$$=\frac{a^2}{b}\int\frac{1+\cos2\theta}{2}\,d\theta=\frac{a^2}{2b}\left(\theta+\frac{\sin2\theta}{2}\right)+C=\frac{a^2}{2b}\left(\sin^{-1}\frac{b}{a}x+\sin\theta\cos\theta\right)+C$$

$$=\frac{a^2}{2b}\left(\sin^{-1}\frac{b}{a}x+\frac{b}{a}x\sqrt{1-\left(\frac{bx}{a}\right)^2}\right)+C=\frac{a^2}{2b}\sin^{-1}\frac{b}{a}x+\frac{x}{2}\sqrt{a^2-b^2x^2}+C$$

$$=\frac{x}{2}\sqrt{a^2-b^2x^2}+\frac{a^2}{2b}\sin^{-1}\frac{bx}{a}+C$$