Rectangular Hyperbola

A rectangular hyperbola is one with both asymptotes meet at right angles. i.e. $y = \pm x$.

$\because$ The asymptotes of a hyperbola is $\dfrac{x}{a} = \pm\dfrac{y}{b}$, $\quad\therefore\quad a = b$, hence the following properties:

When foci are on x-axis:

Cartesian equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{a^2} = 1$ or $\boxed{x^2 - y^2 = a^2}$

Eccentricity: $e = \sqrt{1 + \dfrac{a^2}{a^2}} = \sqrt{2}$

Foci: $S = (a\sqrt{2}, 0), \quad S' = (-a\sqrt{2}, 0)$

Directrices: $m : x = \dfrac{a}{\sqrt{2}}, \quad m' : x = -\dfrac{a}{\sqrt{2}}$

Asymptotes: $x = \pm y$

... similar to those properties of a hyperbola in general.

When the above hyperbola is rotated anti-clockwise by $\frac{\pi}{4}$ on a complex plan:

Before (foci on x-axis): $|z - ae| = e\left[Re(z) - \dfrac{a}{e}\right] = e\left(\dfrac{z+\overline{z}}{2} - \dfrac{a}{e}\right)$, $\quad \left|z - a\sqrt{2}\right| = \sqrt{2}\left(\dfrac{z+\overline{z}}{2} - \dfrac{a}{\sqrt{2}}\right)$

$$\left|z - a\sqrt{2}\right|^2 = 2\left(\frac{z+\overline{z}}{2} - \frac{a}{\sqrt{2}}\right)^2, \quad 2\left(z - a\sqrt{2}\right)\cdot\left(\overline{z} - a\sqrt{2}\right) - 4\left(\frac{z+\overline{z}}{2} - \frac{a}{\sqrt{2}}\right)^2 = 0$$

$$2\left(z\overline{z} - a\sqrt{2}z - a\sqrt{2}\overline{z} + 2a^2\right) - \left(z + \overline{z} - a\sqrt{2}\right)^2 = 0$$

$$\left(2z\overline{z} - 2za\sqrt{2} - 2\overline{z}a\sqrt{2} + 4a^2\right) - \left(z^2 + \overline{z}^2 + 2a^2 + 2z\overline{z} - 2za\sqrt{2} - 2\overline{z}a\sqrt{2}\right) = 0$$

$$2a^2 - z^2 - \overline{z}^2 = 0, \quad \boxed{z^2 + \overline{z}^2 = 2a^2}$$

After (foci on $y = x$): Rotate clockwise by $\frac{\pi}{4}$ will fit the above. i.e. to substitute z by $(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})z = \frac{1}{\sqrt{2}}(1-i)z$.

$$\left[\tfrac{1}{\sqrt{2}}(1-i)z\right]^2 + \overline{\left[\tfrac{1}{\sqrt{2}}(1-i)z\right]}^2 = 2\,a^2$$

$$[(1-i)(x+iy)]^2 + [(1+i)(x-iy)]^2 = 4\,a^2$$

$$[x + y - i(x-y)]^2 + [x + y + i(x-y)]^2 = 4\,a^2$$

$$2\left[(x+y)^2 - (x-y)^2\right] = 4\,a^2, \quad 8\,xy = 4\,a^2, \quad \boxed{xy = \frac{a^2}{2}}$$

Since the foci are on $y = x$, the vertices are $\left(\pm\dfrac{a}{\sqrt{2}}, \pm\dfrac{a}{\sqrt{2}}\right)$ or $(\pm c, \pm c)$, where $\boxed{c = \dfrac{a}{\sqrt{2}}}$.

The Rectangular Hyperbola becomes $\boxed{xy = c^2}$.

Foci rotated by $\frac{\pi}{4}$: When the foci is on x-axis, they are $\pm\, a\sqrt{2}$

When the foci is on $y = x$, they are rotated by $\frac{\pi}{4}$: $\pm\,\frac{1}{\sqrt{2}}(1+i)\cdot a\sqrt{2} = \pm a \pm ia$

Foci: $\boxed{S = (a, a), \quad S' = (-a, -a)}$ or $\boxed{S = (c\sqrt{2}, c\sqrt{2}), \quad S' = (-c\sqrt{2}, -c\sqrt{2})}$

The directrices of a "horizontal" hyperbola on complex plan: $Re(z) = \pm\dfrac{a}{\sqrt{2}}$

The directrices after rotated by $-\frac{\pi}{4}$ will become $Re(\frac{1}{\sqrt{2}}(1-i)z) = \pm\dfrac{a}{\sqrt{2}}$

$$Re[(1-i)(x+iy)] = \pm a, \quad Re[(x+y) - i(x-y)] = \pm a, \quad x + y = \pm a$$

Directrices: $\boxed{x + y = \pm a}$ or $\boxed{x + y = \pm c\sqrt{2}}$

Asymptotes: $\boxed{x = 0, \quad y = 0}$ $\quad$ ($x = \pm y$ rotated by $\frac{\pi}{4}$)

Tangents and Chords: In the following,

$P(x_1, y_1)$ or $P(cp, \frac{c}{p})$ is on the hyperbola, so is

$Q(x_2, y_2)$ or $Q(cq, \frac{c}{q})$, and

$T(x_0, y_0)$ or $T(ct, \frac{c}{t})$ is the intersection of the two tangents from P and Q.

Cartesian Form: $xy = c^2$

Derivative: $y\,dx + x\,dy = 0\,, \quad \dfrac{dy}{dx} = -\dfrac{y}{x}$

Tangent at P: $\dfrac{y - y_1}{x - x_1} = -\dfrac{y_1}{x_1}\,, \quad x_1 y - x_1 y_1 + x y_1 - x_1 y_1 = 0\,, \quad$ (Note: $x_1 y_1 = c^2$)

$$\boxed{y_1 x + x_1 y = 2c^2}$$

Normal at P: $\dfrac{y - y_1}{x - x_1} = \dfrac{x_1}{y_1}\,, \quad y_1 y - y_1^2 - x_1 x + x_1^2 = 0\,, \quad \boxed{x_1 x - y_1 y = x_1^2 - y_1^2}$

Intersection T: $\boxed{(x_0, y_0) = \left[\dfrac{2c^2(x_1 - x_2)}{x_1 y_2 - x_2 y_1}, \dfrac{-2c^2(y_1 - y_2)}{x_1 y_2 - x_2 y_1}\right]}$

Chord of Contact PQ: $\boxed{y_0 x + x_0 y = 2c^2}$ as (x_0, y_0) is on both $\quad y x_1 + x y_1 = 2c^2 \quad$ and $\quad y x_2 + x y_2 = 2c^2$

So $\dfrac{-2c^2(y_1 - y_2)}{x_1 y_2 - x_2 y_1} x + \dfrac{2c^2(x_1 - x_2)}{x_1 y_2 - x_2 y_1} y = 2c^2$

$(y_2 - y_1)x + (x_1 - x_2)y = x_1 y_2 - x_2 y_1$

$\left(\dfrac{c^2}{x_2} - \dfrac{c^2}{x_1}\right) x + (x_1 - x_2)y = x_1 \dfrac{c^2}{x_2} - x_2 \dfrac{c^2}{x_1}$

$\times x_1 x_2:\quad c^2(x_1 - x_2)x + x_1 x_2 (x_1 - x_2) y = c^2(x_1^2 - x_2^2) = c^2(x_1 - x_2)(x_1 + x_2)$

$$\boxed{c^2 x + x_1 x_2 y = c^2(x_1 + x_2)}$$

Parametric Form: $x = ct,\ \ y = \dfrac{c}{t}\,, \quad$ where $t \neq 0$

Derivative: $\dfrac{dy}{dx} = \dfrac{-ct^{-2}}{c} = -\dfrac{1}{t^2}$

Tangent at P: $\dfrac{y - \frac{c}{p}}{x - cp} = -\dfrac{1}{p^2}\,, \quad \boxed{x + p^2 y - 2cp = 0}$

Normal at P: $\dfrac{y - \frac{c}{p}}{x - cp} = p^2\,, \quad p^2(x - cp) - (y - \dfrac{c}{p}) = 0\,,$

$\times \dfrac{1}{p}:\quad px - cp^2 - \dfrac{y}{p} + \dfrac{c}{p^2} = 0\,, \quad \boxed{px - \dfrac{1}{p} y = c\left(p^2 - \dfrac{1}{p^2}\right)}$

Intersection T: $(x_0, y_0) = \left[\dfrac{2c^2(cp - cq)}{cp \cdot \frac{c}{q} - cq \cdot \frac{c}{p}}, \dfrac{-2c^2\left(\frac{c}{p} - \frac{c}{q}\right)}{cp \cdot \frac{c}{q} - cq \cdot \frac{c}{p}}\right]$

$= \left[\dfrac{2c(p - q)}{\frac{p}{q} - \frac{q}{p}}, \dfrac{-2c\left(\frac{q-p}{pq}\right)}{\frac{p}{q} - \frac{q}{p}}\right] = \left[\dfrac{2cpq(p - q)}{p^2 - q^2}, \dfrac{-2c(q - p)}{p^2 - q^2}\right]$

$= \boxed{\left(\dfrac{2cpq}{p + q}, \dfrac{2c}{p + q}\right)}$

Chord of Contact PQ: $\dfrac{2cx}{p + q} + \dfrac{2cpqy}{p + q} = 2c^2$

$$\boxed{x + pqy = c(p + q)}$$