Parabola

In the following, $a \neq 0$,
$P(x_1, y_1)$ or $P(2ap, ap^2)$ is on the parabola, so is
$Q(x_2, y_2)$ or $Q(2aq, aq^2)$, and
$T(x_0, y_0)$ is the intersection of the two tangents from P and Q.

The Basics

When focus is on x-axis:

Cartesian equation:	$y^2 = 4ax$
Parametric equation:	$y = 2at, \quad x = at^2$
Eccentricity:	$e = 1$
x-intercept:	$(0, 0)$
Foci:	$S = (a, 0)$
Directrices:	$m : \ x = -a$
Locus:	$P(x, y) \ : \quad PS = PL \ , \quad$ where $PL \perp m$

When focus is on y-axis:

Cartesian equation:	$x^2 = 4ay$
Parametric equation:	$x = 2at, \quad y = at^2$
Eccentricity:	$e = 1$
y-intercept:	$(0, 0)$
Foci:	$S = (0, a)$
Directrix:	$m : \ y = -a$
Locus:	$P(x, y) \ : \quad PS = PL \ , \quad$ where $PL \perp m$

Tangents and Chords (Cartesian) $x^2 = 4ay$

Derivative:	$\frac{dy}{dx} = \frac{x}{2a}$
Tangent at P:	$x_1 x = 2a(y + y_1)$
Normal at P:	$2ax + x_1 y = 2ax_1 + x_1 y_1$
Intersection T:	$(x_0, y_0) = \left(\frac{2a(y_2 - y_1)}{x_2 - x_1}, \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1} \right)$
Chord of Contact PQ:	$x_0 x = 2a(y + y_0)$
Focal Chord:	T is on directrix $\Leftrightarrow PQ$ is a focal chord

Proof: $y_0 = -a \Leftrightarrow S = (0, a)$ satisfies $x_0 x = 2a(y + y_0)$

Tangents and Chords (Parametric)

Derivative:	$\frac{dy}{dp} = 2ap \ , \quad \frac{dx}{dp} = 2a \ , \frac{dy}{dx} = p$
Tangent at P:	$y = px - ap^2$
Normal at P:	$x + py = 2ap + ap^3$
Intersection T:	$\big(a(p + q), apq\big)$
Chord of Contact PQ:	$y = \frac{1}{2}(p + q)x - apq$
Focal Chord:	$pq = -1 \Leftrightarrow PQ$ is a focal chord

Proof: $pq = -1 \Leftrightarrow S = (0, a)$ satisfies $y = \frac{1}{2}(p + q)x - apq$

Parabola – the case of a hyperbola with $e \to 1^+$

Hyperbola with foci on y-axis, shifted down by b: $\dfrac{(y+b)^2}{b^2} - \dfrac{x^2}{a^2} = 1$

When $2kb = a^2$ $(k \neq 0)$, upper half is $(y \geq 0)$:

$$y = b\left(\sqrt{1+\frac{x^2}{a^2}} - 1\right) = \frac{a^2}{2k}\left(\sqrt{1+\frac{x^2}{a^2}} - 1\right)$$

$$= \frac{a^2}{2k}\left(\sqrt{1+\frac{x^2}{a^2}} - 1\right) \cdot \frac{\sqrt{1+\frac{x^2}{a^2}}+1}{\sqrt{1+\frac{x^2}{a^2}}+1}$$

$$= \frac{a^2}{2k}\frac{\left(1+\frac{x^2}{a^2}\right)-1}{\sqrt{1+\frac{x^2}{a^2}}+1} = \frac{x^2}{2k}\frac{1}{\sqrt{1+\frac{x^2}{a^2}}+1}$$

When $a \to +\infty$: $y = \dfrac{x^2}{4k}$, k becomes the focal length of the parabola.

Eccentricity: $\displaystyle \lim_{a\to\infty} e = \lim_{a\to\infty} \sqrt{1+\frac{a^2}{b^2}} = \lim_{a\to\infty} \sqrt{1+\frac{4k^2}{a^2}} = 1$