Hyperbola

In the following, $a > 0, \quad b > 0,$

$P(x_1, y_1)$ or $P(a\sec\theta_1, b\tan\theta_1)$ is on the hyperbola, so is

$Q(x_2, y_2)$ or $Q(a\sec\theta_2, b\tan\theta_2)$, and

$T(x_0, y_0)$ or $T(a\sec\theta_0, b\tan\theta_0)$ is the intersection of the two tangents from P and Q.

The Basics

When foci are on x-axis:

Cartesian equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Parametric equation: $x = a\sec\theta, \quad y = b\tan\theta$

x-intercept: $A = (a, 0), \quad A' = (-a, 0)$

y-intercept: none

Eccentricity: $e = \sqrt{1 + \dfrac{b^2}{a^2}}\,, \quad e > 1$

Foci: $S = (ae, 0), \quad S' = (-ae, 0)$ (further from the origin than a)

Directrices: $m:\ x = \dfrac{a}{e}, \quad m':\ x = -\dfrac{a}{e}$ (closer to the origin than a)

Locus: $P(x, y)\ : \quad PS - PS' = \pm 2a$

Asymptotes: $\dfrac{x}{a} = \pm\dfrac{y}{b}$

When foci are on y-axis:

Cartesian equation: $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Parametric equation: $y = b\sec\theta, \quad x = a\tan\theta\,,$ where θ starts on +y-axis

or $y = b\sec\left(\theta - \dfrac{\pi}{2}\right), \quad x = a\tan\left(\theta - \dfrac{\pi}{2}\right),$

i.e. $y = b\csc\theta, \quad x = -a\cot\theta\,,$ where θ starts on +x-axis

x-intercept: none

y-intercept: $A = (0, b), \quad A' = (0, -b)$

Eccentricity: $e = \sqrt{1 + \dfrac{a^2}{b^2}}\,, \quad e > 1$

Foci: $S = (0, be), \quad S' = (0, -be)$ (further from the origin than a)

Directrices: $m:\ y = \dfrac{b}{e}, \quad m':\ y = -\dfrac{b}{e}$ (closer to the origin than a)

Locus: $P(x, y)\ : \quad PS - PS' = \pm 2b$

Asymptotes: $\dfrac{x}{a} = \pm\dfrac{y}{b}$

Tangents and Chords (Cartesian) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Derivative: $\dfrac{dy}{dx} = \dfrac{b^2}{a^2}\dfrac{x}{y}$

Tangent at P: $\dfrac{x_1 x}{a^2} - \dfrac{y_1 y}{b^2} = 1$

Normal at P: $\dfrac{a^2 x}{x_1} + \dfrac{b^2 y}{y_1} = a^2 + b^2$

Intersection T: $(x_0, y_0) = \left(\dfrac{a^2(y_2 - y_1)}{x_1 y_2 - x_2 y_1}, \dfrac{b^2(x_2 - x_1)}{x_1 y_2 - x_2 y_1}\right)$

Chord of Contact PQ: $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1$

Focal Chord: T is on directrix $\Leftrightarrow$ PQ is a focal chord

Proof: $x_0 = \pm\dfrac{a}{e} \Leftrightarrow (S, 0) = (\pm ae, 0)$ satisfies $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1$

as $\dfrac{\pm\frac{a}{e} \cdot \pm ae}{a^2} - \dfrac{0}{b^2} = 1$

Tangents and Chords (Parametric) $x = a\sec\theta,\ \ y = b\tan\theta$

Derivative: $\dfrac{dy}{dx} = \dfrac{b\sec\theta}{a\tan\theta}$

Tangent at P: $\dfrac{x\sec\theta_1}{a} - \dfrac{y\tan\theta_1}{b} = 1$

Normal at P: $\dfrac{ax}{\sec\theta_1} + \dfrac{by}{\tan\theta_1} = a^2 + b^2$

Intersection T: $\left(\dfrac{a \cdot \cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)},\ \dfrac{b \cdot \sin\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}\right)$

$= \left(\frac{a\sin(\theta_1 - \theta_2)}{\sin\theta_1 - \sin\theta_2}, \frac{-b(\cos\theta_1 - \cos\theta_2)}{\sin\theta_1 - \sin\theta_2}\right)$

Chord of Contact PQ: $\dfrac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \dfrac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$

$\frac{x}{a} \cdot \frac{\sin(\theta_1 - \theta_2)}{\sin\theta_1 - \sin\theta_2} + \frac{y}{b} \cdot \frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_1 - \sin\theta_2} = 1$