Hyperbola – The Working Out

In the following, $a > 0, \quad b > 0,$
$P(x_1, y_1)$ or $P(a\sec\theta_1, b\tan\theta_1)$ is on the hyperbola, so is
$Q(x_2, y_2)$ or $Q(a\sec\theta_2, b\tan\theta_2)$, and
$T(x_0, y_0)$ or $T(a\sec\theta_0, b\tan\theta_0)$ is the intersection of the two tangents from P and Q.

Hyperbola (Cartesian) $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1\,, \quad b^2x^2 - a^2y^2 = a^2b^2\,, \quad y = \pm b\sqrt{\dfrac{x^2}{a^2} - 1}$

Derivative: $\mathbf{\dfrac{b^2}{a^2}\dfrac{x}{y}}$

$$b^2x^2 - a^2y^2 = a^2b^2$$
$$2b^2x\,dx - 2a^2y\,dy = 0$$
$$\not{2}a^2y\,dy = \not{2}b^2x\,dx$$
$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Tangent at P: $\mathbf{\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1}$

$$y - y_1 = \left(\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$$
$$\times\frac{y_1}{b^2}: \quad \frac{y_1}{b^2}y - \frac{y_1}{b^2}y_1 = \frac{x_1}{a^2}x - \frac{x_1}{a^2}x_1$$
$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

Normal at P: $\mathbf{\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2}$

Gradient is $\quad -\dfrac{a^2}{b^2}\dfrac{y_1}{x_1}$

$$y - y_1 = \left(-\frac{a^2}{b^2}\frac{y_1}{x_1}\right)(x - x_1)$$
$$\times\frac{b^2}{y_1}: \quad \frac{b^2}{y_1}y - \frac{b^2}{\not{y_1}}\not{y_1} = -\frac{a^2}{x_1}x + \frac{a^2}{\not{x_1}}\not{x_1}$$
$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

$$\text{Intersection } T\text{:} \quad (\mathbf{x_0}, \mathbf{y_0}) = \left(\frac{\mathbf{a^2(y_2 - y_1)}}{\mathbf{x_1y_2 - x_2y_1}}, \frac{\mathbf{b^2(x_2 - x_1)}}{\mathbf{x_1y_2 - x_2y_1}} \right)$$

$$PT: \quad \frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1 \tag{1}$$

$$QT: \quad \frac{x_2x}{a^2} - \frac{y_2y}{b^2} = 1 \tag{2}$$

$$(1) \times y_2 - (2) \times y_1: \quad \frac{x_1y_2x}{a^2} - \frac{x_2y_1x}{a^2} = y_2 - y_1$$

$$(x_1y_2 - x_2y_1)x = a^2(y_2 - y_1)$$

$$x = \frac{a^2(y_2 - y_1)}{x_1y_2 - x_2y_1}$$

$$(1) \times x_2 - (2) \times x_1: \quad -\frac{x_2y_1y}{b^2} + \frac{x_1y_2y}{b^2} = x_2 - x_1$$

$$(x_1y_2 - x_2y_1)y = b^2(x_2 - x_1)$$

$$y = \frac{b^2(x_2 - x_1)}{x_1y_2 - x_2y_1}$$

$$\text{Chord of Contact } PQ\text{:} \quad \frac{\mathbf{x_0x}}{\mathbf{a^2}} - \frac{\mathbf{y_0y}}{\mathbf{b^2}} = \mathbf{1}$$

$$PT: \quad \frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$$

$$QT: \quad \frac{x_2x}{a^2} - \frac{y_2y}{b^2} = 1$$

$T(x_0, y_0)$ is on both PT and QT, so

$$\frac{x_1x_0}{a^2} - \frac{y_1y_0}{b^2} = 1$$

$$\frac{x_2x_0}{a^2} - \frac{y_2y_0}{b^2} = 1$$

$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$ satisfies both P and Q,

and therefore is the Chord of Contact.

Hyperbola (Parametric) $\quad x = a\sec\theta, \; y = b\tan\theta$

$$\text{Derivative:} \quad \frac{\mathbf{b}\sec\theta}{\mathbf{a}\tan\theta}$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta\,, \qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$$

$$\text{Tangent at } P\text{:} \quad \frac{\mathbf{x}\sec\theta_\mathbf{1}}{\mathbf{a}} - \frac{\mathbf{y}\tan\theta_\mathbf{1}}{\mathbf{b}} = \mathbf{1}$$

$$y - y_1 = \frac{b\sec\theta_1}{a\tan\theta_1}(x - x_1)$$

$$\times\frac{\tan\theta_1}{b}: \quad \frac{\tan\theta_1}{b}(y - b\tan\theta_1) = \frac{\sec\theta_1}{a}(x - a\sec\theta_1)$$

$$\frac{y\tan\theta_1}{b} - \tan^2\theta_1 = \frac{x\sec\theta_1}{a} - \sec^2\theta_1$$

$$\frac{x\sec\theta_1}{a} - \frac{y\tan\theta_1}{b} = \sec^2\theta_1 - \tan^2\theta_1 = 1$$

$$
\begin{aligned}
\text{Normal at } P\text{:} \quad & \frac{\mathbf{ax}}{\sec\theta_{\mathbf{1}}} + \frac{\mathbf{by}}{\tan\theta_{\mathbf{1}}} = \mathbf{a^2} + \mathbf{b^2} \\
& y - y_1 = -\frac{a\tan\theta_1}{b\sec\theta_1}(x - x_1) \\
\times\frac{b}{\tan\theta_1}: \quad & \frac{b}{\tan\theta_1}(y - b\tan\theta_1) = -\frac{a}{\sec\theta_1}(x - a\sec\theta_1) \\
& \frac{by}{\tan\theta_1} - b^2 = -\frac{ax}{\sec\theta_1} + a^2 \\
& \frac{ax}{\sec\theta_1} + \frac{by}{\tan\theta_1} = a^2 + b^2
\end{aligned}
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$$
\begin{aligned}
\text{Intersection } T\text{:} \quad & \left(\frac{\mathbf{a}\sin(\theta_{\mathbf{1}} - \theta_{\mathbf{2}})}{\sin\theta_{\mathbf{1}} - \sin\theta_{\mathbf{2}}}, \frac{-\mathbf{b}(\cos\theta_{\mathbf{1}} - \cos\theta_{\mathbf{2}})}{\sin\theta_{\mathbf{1}} - \sin\theta_{\mathbf{2}}}\right) \\
PT: \quad & \frac{x\sec\theta_1}{a} - \frac{y\tan\theta_1}{b} = 1 \qquad (1) \\
QT: \quad & \frac{x\sec\theta_2}{a} - \frac{y\tan\theta_2}{b} = 1 \qquad (2) \\
(1)\times\tan\theta_2 - (2)\times\tan\theta_1: \quad & \frac{x}{a}(\sec\theta_1\tan\theta_2 - \tan\theta_1\sec\theta_2) = \tan\theta_2 - \tan\theta_1 \\
& \frac{x}{a}\left(\frac{1}{\cos\theta_1}\cdot\frac{\sin\theta_2}{\cos\theta_2} - \frac{\sin\theta_1}{\cos\theta_1}\cdot\frac{1}{\cos\theta_2}\right) = \frac{\sin\theta_2}{\cos\theta_2} - \frac{\sin\theta_1}{\cos\theta_1} \\
& \frac{x}{a}\left(\frac{\sin\theta_2 - \sin\theta_1}{\cos\theta_1\cos\theta_2}\right) = \frac{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2} = \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1\cos\theta_2} \\
& x = \frac{a\sin(\theta_1 - \theta_2)}{\sin\theta_1 - \sin\theta_2} \\
(1)\times\sec\theta_2 - (2)\times\sec\theta_1: \quad & -\frac{y}{b}(\sec\theta_2\tan\theta_1 - \tan\theta_2\sec\theta_1) = \sec\theta_2 - \sec\theta_1 \\
& -\frac{y}{b}\left(\frac{1}{\cos\theta_2}\cdot\frac{\sin\theta_1}{\cos\theta_1} - \frac{\sin\theta_2}{\cos\theta_2}\cdot\frac{1}{\cos\theta_1}\right) = \frac{1}{\cos\theta_2} - \frac{1}{\cos\theta_1} \\
& -\frac{y}{b}\left(\frac{\sin\theta_1 - \sin\theta_2}{\cos\theta_2\cos\theta_1}\right) = \frac{\cos\theta_1 - \cos\theta_2}{\cos\theta_2\cos\theta_1} \\
& y = \frac{-b(\cos\theta_1 - \cos\theta_2)}{\sin\theta_1 - \sin\theta_2}
\end{aligned}
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Chord of Contact PQ: $\quad \frac{\mathbf{x}}{\mathbf{a}}\cos\left(\frac{\boldsymbol{\theta_1-\theta_2}}{\mathbf{2}}\right)-\frac{\mathbf{y}}{\mathbf{b}}\sin\left(\frac{\boldsymbol{\theta_1+\theta_2}}{\mathbf{2}}\right)=\cos\left(\frac{\boldsymbol{\theta_1+\theta_2}}{\mathbf{2}}\right)$

First Prove: $\quad \frac{x}{a}\cdot\frac{\sin(\theta_1-\theta_2)}{\sin\theta_1-\sin\theta_2}+\frac{y}{b}\cdot\frac{\cos\theta_1-\cos\theta_2}{\sin\theta_1-\sin\theta_2}=1$

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\begin{aligned}
PQ: \quad & \frac{y-b\tan\theta_1}{x-a\sec\theta_1}=\frac{b\tan\theta_1-b\tan\theta_2}{a\sec\theta_1-a\sec\theta_2}\\
& (y-b\tan\theta_1)\cdot(a\sec\theta_1-a\sec\theta_2)=(x-a\sec\theta_1)\cdot(b\tan\theta_1-b\tan\theta_2)\\
& ay(\sec\theta_1-\sec\theta_2)-ab\tan\theta_1(\sec\theta_1-\sec\theta_2)=bx(\tan\theta_1-\tan\theta_2)-ab\sec\theta_1(\tan\theta_1-\tan\theta_2)\\
& ay(\sec\theta_1-\sec\theta_2)-bx(\tan\theta_1-\tan\theta_2)=ab\tan\theta_1(\sec\theta_1-\sec\theta_2)-ab\sec\theta_1(\tan\theta_1-\tan\theta_2)\\
\times\frac{1}{ab}: \quad & \frac{y}{b}(\sec\theta_1-\sec\theta_2)-\frac{x}{a}(\tan\theta_1-\tan\theta_2)=\tan\theta_1(\sec\theta_1-\sec\theta_2)-\sec\theta_1(\tan\theta_1-\tan\theta_2)\\
& \frac{y}{b}(\sec\theta_1-\sec\theta_2)-\frac{x}{a}(\tan\theta_1-\tan\theta_2)=-\tan\theta_1\sec\theta_2+\sec\theta_1\tan\theta_2\\
& \frac{y}{b}\left(\frac{1}{\cos\theta_1}-\frac{1}{\cos\theta_2}\right)-\frac{x}{a}\left(\frac{\sin\theta_1}{\cos\theta_1}-\frac{\sin\theta_2}{\cos\theta_2}\right)=-\frac{\sin\theta_1}{\cos\theta_1}\cdot\frac{1}{\cos\theta_2}+\frac{1}{\cos\theta_1}\cdot\frac{\sin\theta_2}{\cos\theta_2}\\
& \frac{y}{b}\left(\frac{\cos\theta_2-\cos\theta_1}{\cos\theta_1\cos\theta_2}\right)-\frac{x}{a}\left(\frac{\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}\right)=\frac{\sin\theta_2-\sin\theta_1}{\cos\theta_1\cos\theta_2}\\
& \frac{x}{a}\left(\frac{\sin(\theta_1-\theta_2)}{\cos\theta_1\cos\theta_2}\right)-\frac{y}{b}\left(\frac{\cos\theta_2-\cos\theta_1}{\cos\theta_1\cos\theta_2}\right)=\frac{\sin\theta_1-\sin\theta_2}{\cos\theta_1\cos\theta_2}\\
\therefore \quad & \frac{x}{a}\cdot\frac{\sin(\theta_1-\theta_2)}{\sin\theta_1-\sin\theta_2}+\frac{y}{b}\cdot\frac{\cos\theta_1-\cos\theta_2}{\sin\theta_1-\sin\theta_2}=1\\
& \frac{x}{a}\cdot\frac{\cancel{2}\,\cancel{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cancel{2}\cos\left(\frac{\theta_1+\theta_2}{2}\right)\,\cancel{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)}+\frac{y}{b}\cdot\frac{-\cancel{2}\sin\left(\frac{\theta_1+\theta_2}{2}\right)\,\cancel{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)}{\cancel{2}\cos\left(\frac{\theta_1+\theta_2}{2}\right)\,\cancel{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)}=1\\
& \frac{x}{a}\cdot\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}-\frac{y}{b}\cdot\frac{\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}=1\\
\therefore \quad & \frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right)
\end{aligned}
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