Ellipse

In the following, $a > 0, \quad b > 0,$

$P(x_1, y_1)$ or $P(a\cos\theta_1, b\sin\theta_1)$ is on the ellipse, so is
$Q(x_2, y_2)$ or $Q(a\cos\theta_2, b\sin\theta_2)$, and
$T(x_0, y_0)$ or $T(a\cos\theta_0, b\sin\theta_0)$ is the intersection of the two tangents from P and Q.

The Basics

Cartesian equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Parametric equation: $x = a\cos\theta, \quad y = b\sin\theta$

When $a > b$, long axis and therefore foci are on x-axis.

x-intercept: $A = (a, 0), \quad A' = (-a, 0)$

y-intercept: $B = (0, b), \quad B' = (0, -b)$

Eccentricity: $e = \sqrt{1 - \dfrac{b^2}{a^2}}, \quad 0 < e < 1$

Foci: $S = (ae, 0), \quad S' = (-ae, 0)$ (closer to the origin than a)

Directrices: $m : x = \dfrac{a}{e}, \quad m' : x = -\dfrac{a}{e}$ (further from the origin than a)

Locus: $P(x, y) : \quad PS + PS' = 2a$

When $b > a$, long axis and therefore foci are on y-axis.

x-intercept: $A = (a, 0), \quad A' = (-a, 0)$

y-intercept: $B = (0, b), \quad B' = (0, -b)$

Eccentricity: $e = \sqrt{1 - \dfrac{a^2}{b^2}}, \quad 0 < e < 1$

Foci: $S = (0, be), \quad S' = (0, -be)$ (closer to the origin than b)

Directrices: $m : y = \dfrac{b}{e}, \quad m' : y = -\dfrac{b}{e}$ (further from the origin than b)

Locus: $P(x, y) : \quad PS + PS' = 2b$

When $a = b = r$, a circle!

x-intercept: $A = (r, 0), \quad A' = (-r, 0)$

y-intercept: $B = (0, r), \quad B' = (0, -r)$

Eccentricity: $e = 0 \quad \left(= \sqrt{1 - \dfrac{r^2}{r^2}} \right)$

Centre: $(0, 0)$ (Two foci coincide.)

Directrices: Do not exist (at infinity)

Locus: $P(x, y) : \quad PO = r$

Tangents and Chords (Cartesian) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Derivative: $\dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}$

Tangent at P: $\dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 1$

Normal at P: $\dfrac{a^2 x}{x_1} - \dfrac{b^2 y}{y_1} = a^2 - b^2$

Intersection T: $(x_0, y_0) = \left(\dfrac{a^2(y_2 - y_1)}{x_1 y_2 - x_2 y_1}, \dfrac{-b^2(x_2 - x_1)}{x_1 y_2 - x_2 y_1} \right)$

Chord of Contact PQ: $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} = 1$

Focal Chord: T is on directrix $\Leftrightarrow$ PQ is a focal chord

Proof: $x_0 = \pm\dfrac{a}{e} \Leftrightarrow S = (\pm ae, 0)$ satisfies $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1$

as $\dfrac{\pm\frac{a}{e} \cdot \pm ae}{a^2} - \dfrac{0}{b^2} = 1$

Tangents and Chords (Parametric) $x = a\cos\theta,\ y = b\sin\theta$

Derivative: $\dfrac{dy}{dx} = -\dfrac{b\cos\theta}{a\sin\theta}$

Tangent at P: $\dfrac{x\cos\theta_1}{a} + \dfrac{y\sin\theta_1}{b} = 1$

Normal at P: $\dfrac{ax}{\cos\theta_1} - \dfrac{by}{\sin\theta_1} = a^2 - b^2$

Intersection T: $\left(\dfrac{a \cdot \cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \dfrac{b \cdot \sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} \right)$

$= \left(\frac{a(\sin\theta_1 - \sin\theta_2)}{\sin(\theta_1 - \theta_2)}, \frac{-b(\cos\theta_1 - \cos\theta_2)}{\sin(\theta_1 - \theta_2)} \right)$

Chord of Contact PQ: $\dfrac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right) + \dfrac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1-\theta_2}{2}\right)$

$\frac{x}{a} \cdot \frac{\sin\theta_1 - \sin\theta_2}{\sin(\theta_1 - \theta_2)} - \frac{y}{b} \cdot \frac{\cos\theta_1 - \cos\theta_2}{\sin(\theta_1 - \theta_2)} = 1$