Ellipse – The Working Out

In the following, $a > 0, \quad b > 0,$

$P(x_1, y_1)$ or $P(a\cos\theta_1, b\sin\theta_1)$ is on the ellipse, so is

$Q(x_2, y_2)$ or $Q(a\cos\theta_2, b\sin\theta_2)$, and

$T(x_0, y_0)$ or $T(a\cos\theta_0, b\sin\theta_0)$ is the intersection of the two tangents from P and Q.

Ellipse (Cartesian) $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\,, \quad b^2x^2 + a^2y^2 = a^2b^2\,, \quad y = \pm b\sqrt{1 - \dfrac{x^2}{a^2}}$

Derivative: $\quad -\dfrac{\mathbf{b^2}}{\mathbf{a^2}}\dfrac{\mathbf{x}}{\mathbf{y}}$

$$b^2x^2 + a^2y^2 = a^2b^2$$
$$2b^2x\,dx + 2a^2y\,dy = 0$$
$$\not{2}a^2y\,dy = -\not{2}b^2x\,dx$$
$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Tangent at P: $\quad \dfrac{\mathbf{x_1x}}{\mathbf{a^2}} + \dfrac{\mathbf{y_1y}}{\mathbf{b^2}} = \mathbf{1}$

$$y - y_1 = \left(-\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$$
$$\times\frac{y_1}{b^2}: \quad \frac{y_1}{b^2}y - \frac{y_1}{b^2}y_1 = -\frac{x_1}{a^2}x + \frac{x_1}{a^2}x_1$$
$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

Normal at P: $\quad \dfrac{\mathbf{a^2x}}{\mathbf{x_1}} - \dfrac{\mathbf{b^2y}}{\mathbf{y_1}} = \mathbf{a^2 - b^2}$

Gradient is $\quad \dfrac{a^2}{b^2}\dfrac{y_1}{x_1}$

$$y - y_1 = \left(\frac{a^2}{b^2}\frac{y_1}{x_1}\right)(x - x_1)$$
$$\times\frac{b^2}{y_1}: \quad \frac{b^2}{y_1}y - \frac{b^2}{\not{y_1}}\not{y_1} = \frac{a^2}{x_1}x - \frac{a^2}{\not{x_1}}\not{x_1}$$
$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

$$\text{Intersection } T: \quad (\mathbf{x_0}, \mathbf{y_0}) = \left(\frac{\mathbf{a^2(y_2 - y_1)}}{\mathbf{x_1 y_2 - x_2 y_1}}, \frac{\mathbf{-b^2(x_2 - x_1)}}{\mathbf{x_1 y_2 - x_2 y_1}} \right)$$

$$PT: \quad \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1 \qquad (1)$$

$$QT: \quad \frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} = 1 \qquad (2)$$

$$(1) \times y_2 - (2) \times y_1: \quad \frac{x_1 y_2 x}{a^2} - \frac{x_2 y_1 x}{a^2} = y_2 - y_1$$

$$(x_1 y_2 - x_2 y_1)x = a^2(y_2 - y_1)$$

$$x = \frac{a^2(y_2 - y_1)}{x_1 y_2 - x_2 y_1}$$

$$(1) \times x_2 - (2) \times x_1: \quad \frac{x_2 y_1 y}{b^2} - \frac{x_1 y_2 y}{b^2} = x_2 - x_1$$

$$-(x_1 y_2 - x_2 y_1)y = b^2(x_2 - x_1)$$

$$y = \frac{-b^2(x_2 - x_1)}{x_1 y_2 - x_2 y_1}$$

$$\text{Chord of Contact } PQ: \quad \frac{\mathbf{x_0 x}}{\mathbf{a^2}} + \frac{\mathbf{y_0 y}}{\mathbf{b^2}} = \mathbf{1}$$

$$PT: \quad \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$$

$$QT: \quad \frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} = 1$$

$T(x_0, y_0)$ is on both PT and QT, so

$$\frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} = 1$$

$$\frac{x_2 x_0}{a^2} + \frac{y_2 y_0}{b^2} = 1$$

$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$ is satisfied by both P and Q,

and therefore is the Chord of Contact.

Ellipse (Parametric) $\quad x = a\cos\theta,\ y = b\sin\theta$

$$\text{Derivative:} \quad -\frac{\mathbf{b}\cos\theta}{\mathbf{a}\sin\theta}$$

$$\frac{dx}{d\theta} = -a\sin\theta\,, \quad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$$

$$\text{Tangent at } P: \quad \frac{\mathbf{x}\cos\theta_{\mathbf{1}}}{\mathbf{a}} + \frac{\mathbf{y}\sin\theta_{\mathbf{1}}}{\mathbf{b}} = \mathbf{1}$$

$$y - y_1 = -\frac{b\cos\theta_1}{a\sin\theta_1}(x - x_1)$$

$$\times\frac{\sin\theta_1}{b}: \quad \frac{\sin\theta_1}{b}(y - b\sin\theta_1) = -\frac{\cos\theta_1}{a}(x - a\cos\theta_1)$$

$$\frac{y\sin\theta_1}{b} - \sin^2\theta_1 = -\frac{x\cos\theta_1}{a} + \cos^2\theta_1$$

$$\frac{x\cos\theta_1}{a} + \frac{y\sin\theta_1}{b} = \sin^2\theta_1 + \cos^2\theta_1 = 1$$

$$\text{Normal at } P: \quad \frac{\mathbf{ax}}{\cos\theta_{\mathbf{1}}} - \frac{\mathbf{by}}{\sin\theta_{\mathbf{1}}} = \mathbf{a^2} - \mathbf{b^2}$$

$$y - y_1 = \frac{a\sin\theta_1}{b\cos\theta_1}(x - x_1)$$

$$\times\frac{b}{\sin\theta_1}: \quad \frac{b}{\sin\theta_1}(y - b\sin\theta_1) = \frac{a}{\cos\theta_1}(x - a\cos\theta_1)$$

$$\frac{by}{\sin\theta_1} - b^2 = \frac{ax}{\cos\theta_1} - a^2$$

$$\frac{ax}{\cos\theta_1} - \frac{by}{\sin\theta_1} = a^2 - b^2$$

$$\text{Intersection } T: \quad \left(\frac{\mathbf{a}(\sin\theta_{\mathbf{1}} - \sin\theta_{\mathbf{2}})}{\sin(\theta_{\mathbf{1}} - \theta_{\mathbf{2}})}, \frac{-\mathbf{b}(\cos\theta_{\mathbf{1}} - \cos\theta_{\mathbf{2}})}{\sin(\theta_{\mathbf{1}} - \theta_{\mathbf{2}})}\right)$$

$$PT: \quad \frac{x\cos\theta_1}{a} + \frac{y\sin\theta_1}{b} = 1 \tag{1}$$

$$QT: \quad \frac{x\cos\theta_2}{a} + \frac{y\sin\theta_2}{b} = 1 \tag{2}$$

$$(1)\times\sin\theta_2 - (2)\times\sin\theta_1: \quad \frac{x}{a}(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2) = \sin\theta_2 - \sin\theta_1$$

$$-\frac{x}{a}\sin(\theta_1 - \theta_2) = -(\sin\theta_1 - \sin\theta_2)$$

$$x = \frac{a(\sin\theta_1 - \sin\theta_2)}{\sin(\theta_1 - \theta_2)}$$

$$(1)\times\cos\theta_2 - (2)\times\cos\theta_1: \quad \frac{y}{b}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) = \cos\theta_2 - \cos\theta_1$$

$$\frac{y}{b}\sin(\theta_1 - \sin\theta_2) = -(\cos\theta_1 - \cos\theta_2)$$

$$y = \frac{-b(\cos\theta_1 - \cos\theta_2)}{\sin(\theta_1 - \theta_2)}$$

$$\text{Chord of Contact } PQ: \quad \frac{\mathbf{x}}{\mathbf{a}}\cos\left(\tfrac{\theta_{\mathbf{1}}+\theta_{\mathbf{2}}}{\mathbf{2}}\right) + \frac{\mathbf{y}}{\mathbf{b}}\sin\left(\tfrac{\theta_{\mathbf{1}}+\theta_{\mathbf{2}}}{\mathbf{2}}\right) = \cos\left(\tfrac{\theta_{\mathbf{1}}-\theta_{\mathbf{2}}}{\mathbf{2}}\right)$$

$$\text{First Prove:} \quad \frac{x}{a}\cdot\frac{\sin\theta_1 - \sin\theta_2}{\sin(\theta_1 - \theta_2)} - \frac{y}{b}\cdot\frac{\cos\theta_1 - \cos\theta_2}{\sin(\theta_1 - \theta_2)} = 1$$

$$PQ: \quad \frac{y - b\sin\theta_1}{x - a\cos\theta_1} = \frac{b\sin\theta_1 - b\sin\theta_2}{a\cos\theta_1 - a\cos\theta_2}$$

$$(y - b\sin\theta_1)\cdot(a\cos\theta_1 - a\cos\theta_2) = (x - a\cos\theta_1)\cdot(b\sin\theta_1 - b\sin\theta_2)$$

$$ay(\cos\theta_1 - \cos\theta_2) - ab\sin\theta_1(\cos\theta_1 - \cos\theta_2) = bx(\sin\theta_1 - \sin\theta_2) - ab\cos\theta_1(\sin\theta_1 - \sin\theta_2)$$

$$ay(\cos\theta_1 - \cos\theta_2) - bx(\sin\theta_1 - \sin\theta_2) = ab\sin\theta_1(\cos\theta_1 - \cos\theta_2) - ab\cos\theta_1(\sin\theta_1 - \sin\theta_2)$$

$$\times\frac{1}{ab}: \quad \frac{y}{b}(\cos\theta_1 - \cos\theta_2) - \frac{x}{a}(\sin\theta_1 - \sin\theta_2) = \sin\theta_1(\cos\theta_1 - \cos\theta_2) - \cos\theta_1(\sin\theta_1 - \sin\theta_2)$$

$$\frac{x}{a}(\sin\theta_1 - \sin\theta_2) - \frac{y}{b}(\cos\theta_1 - \cos\theta_2) = \cos\theta_1(\sin\theta_1 - \sin\theta_2) - \sin\theta_1(\cos\theta_1 - \cos\theta_2)$$

$$\frac{x}{a}(\sin\theta_1 - \sin\theta_2) - \frac{y}{b}(\cos\theta_1 - \cos\theta_2) = -\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 = \sin(\theta_1 - \theta_2)$$

$$\therefore \quad \frac{x}{a}\cdot\frac{\sin\theta_1 - \sin\theta_2}{\sin(\theta_1 - \theta_2)} - \frac{y}{b}\cdot\frac{\cos\theta_1 - \cos\theta_2}{\sin(\theta_1 - \theta_2)} = 1$$

$$\frac{x}{a}\cdot\frac{\not{2}\cos\left(\frac{\theta_1+\theta_2}{2}\right)\not{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)}{\not{2}\not{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)} - \frac{y}{b}\cdot\frac{-\not{2}\sin\left(\frac{\theta_1+\theta_2}{2}\right)\not{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)}{\not{2}\not{\sin}\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)} = 1$$

$$\frac{x}{a}\cdot\frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} + \frac{y}{b}\cdot\frac{\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} = 1$$

$$\therefore \quad \frac{x}{a}\cos\left(\tfrac{\theta_1+\theta_2}{2}\right) + \frac{y}{b}\sin\left(\tfrac{\theta_1+\theta_2}{2}\right) = \cos\left(\tfrac{\theta_1-\theta_2}{2}\right)$$