Ellipse Basics

Foci and Directrices

An ellipse centred at (0,0) with its long axis along the x-axis and its short axis along the y-axis has the following equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } a > b.$$

Let the x-intercepts of the ellipse be $A = (a, 0)$ and $A' = (-a, 0)$, the right focus $S = (k, 0)$ and the right directrix $x = c$ intersecting the x-axis at $L = (c, 0)$.

The eccentricity e of the ellipse is the constant ratio of $SP : PM$ where $PM \perp x = c$ (the directrix) for any P on the ellipse.

For point A, $e = \dfrac{SA}{AM_A} = \dfrac{a-k}{c-a}, \quad a - k = ec - ae \quad \ldots(1).$

For point A', $e = \dfrac{SA'}{A'M_{A'}} = \dfrac{a+k}{c+a}, \quad a + k = ec + ae \quad \ldots(2).$

$(1) + (2): \quad 2a = 2ec, \quad c = \dfrac{a}{e}.$ Similarly for the left directrix. So the directrices are $\boxed{x = \pm\dfrac{a}{e}}$.

$(2) - (1): \quad 2k = 2ae, \quad k = ae.$ Similarly for the left focus. So the foci are $\boxed{S = (ae, 0),\ S' = (-ae, 0)}$.

For a point P on the ellipse directly above S, $x = k$ and $y = \sqrt{b^2\left(1 - \dfrac{k^2}{a^2}\right)} = \sqrt{b^2\left(1 - \dfrac{a^2e^2}{a^2}\right)} = \sqrt{b^2\left(1 - e^2\right)}$.

$$e = \frac{SP}{PM} = \frac{y}{c-k} = \frac{\sqrt{b^2\left(1-e^2\right)}}{\frac{a}{e} - ae}, \quad a - ae^2 = \sqrt{b^2\left(1-e^2\right)}, \quad a^2(1-e^2)^2 = b^2(1-e^2), \quad 1 - e^2 = \frac{b^2}{a^2}, \quad e^2 = 1 - \frac{b^2}{a^2} > 0.$$

$$\boxed{e = \sqrt{1 - \frac{b^2}{a^2}}. \quad 0 < e < 1.}$$