Trigonimetrical Multi-angle Expansion using Complex Numbers

In this context, $z = \cos\theta + i\sin\theta$, $\therefore z^n = \cos n\theta + i\sin n\theta$, where $n \in \mathbb{R}+$

$$z^n = (\cos\theta + i\sin\theta)^n = \sum_{r=0}^{n} C_r^n \cos^{n-r}\theta \cdot i^r \sin^r\theta$$

Ways to remember the following formulae:

1. For cos, coefficents are the *odd* terms of the Pascal's Triangle; for sin, they are the *even* ones. (cos: 1-1, 1-3, 1-6-1, 1-10-5, 1-15-15-1, 1-21-35-7, ...; sin: 2, 3-1, 4-4, 5-10-1, 6-20-6, 7-35-21-1, ...)
2. The odd terms are positive; the even terms are negative. $(+ - + - \ldots)$
3. For cos, first term is $\cos^n\theta$ (or $\cos^n\theta\,\sin^0\theta$); for sin, first term is $\cos^{n-1}\theta\,\sin\theta$.
4. After the first term, the index of cos decreases by 2 and that of sin increases by 2 each term on.
5. For tan, it is a ratio of $\frac{\sin n\theta}{\cos n\theta}$ with a factor of $\cos^n\theta$ taken out from each term to make it $\tan^k\theta$.

When n is odd:

$$\boxed{\cos n\theta = \sum_{s=0}^{k}(-1)^s C_{2s}^n \cos^{n-2s}\theta \cdot \sin^{2s}\theta\,, \quad \text{where } n = 2k+1\ .}$$

e.g. $n = 3, k = 1:$ $\cos 3\theta = \sum_{s=0}^{1}(-1)^s C_{2s}^3 \cos^{3-2s}\theta \cdot \sin^{2s}\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta$

$n = 5, k = 2:$ $\cos 5\theta = \sum_{s=0}^{2}(-1)^s C_{2s}^5 \cos^{5-2s}\theta \cdot \sin^{2s}\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$$\boxed{\sin n\theta = \sum_{s=0}^{k}(-1)^s C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta\,, \quad \text{where } n = 2k+1\ .}$$

e.g. $n = 3, k = 1:$ $\sin 3\theta = \sum_{s=0}^{1}(-1)^s C_{2s+1}^3 \cos^{3-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$

$n = 5, k = 2:$ $\sin 5\theta = \sum_{s=0}^{2}(-1)^s C_{2s+1}^5 \cos^{5-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 5\cos^4\theta \cdot \sin\theta - 10\cos^2\theta \cdot \sin^3\theta + \sin^5\theta$

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

$$\boxed{\tan n\theta = \frac{\sum_{s=0}^{k}(-1)^s C_{2s+1}^n \tan^{2s+1}\theta}{\sum_{s=0}^{k}(-1)^s C_{2s}^n \tan^{2s}\theta}\,, \quad \text{where } n = 2k+1\ .}$$

e.g. $n = 3, k = 1:$ $\tan 3\theta = \frac{\sum_{s=0}^{1}(-1)^s C_{2s+1}^3 \tan^{2s+1}\theta}{\sum_{s=0}^{1}(-1)^s C_{2s}^3 \tan^{2s}\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$n = 5, k = 2:$ $\tan 5\theta = \frac{\sum_{s=0}^{2}(-1)^s C_{2s+1}^5 \tan^{2s+1}\theta}{\sum_{s=0}^{2}(-1)^s C_{2s}^5 \tan^{2s}\theta} = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

When n is even:

$$\boxed{\cos n\theta = \sum_{s=0}^{k} (-1)^s C_{2s}^{n} \cos^{n-2s}\theta \cdot \sin^{2s}\theta\,, \quad \text{where } n = 2k\ .}$$

e.g. $n=2, k=1:$ $\quad \cos 2\theta = \sum_{s=0}^{1} (-1)^s C_{2s}^{2} \cos^{2-2s}\theta \cdot \sin^{2s}\theta = \cos^2\theta - \sin^2\theta$

$n=4, k=2:$ $\quad \cos 4\theta = \sum_{s=0}^{2} (-1)^s C_{2s}^{4} \cos^{4-2s}\theta \cdot \sin^{2s}\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$$\boxed{\sin n\theta = \sum_{s=0}^{k-1} (-1)^s C_{2s+1}^{n} \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta\,, \quad \text{where } n = 2k\ .}$$

e.g. $n=2, k=1:$ $\quad \sin 2\theta = \sum_{s=0}^{0} (-1)^s C_{2s+1}^{2} \cos^{2-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 2\cos\theta\sin\theta$

$n=4, k=2:$ $\quad \sin 4\theta = \sum_{s=0}^{1} (-1)^s C_{2s+1}^{4} \cos^{4-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$$= 4\cos\theta\sin\theta\,(\cos^2\theta - \sin^2\theta) \qquad (= 2\cdot\sin 2\theta\cdot\cos 2\theta)$$

$$\therefore \quad \boxed{\tan n\theta = \frac{\sum_{s=0}^{k-1} (-1)^s C_{2s+1}^{n} \tan^{2s+1}\theta}{\sum_{s=0}^{k} (-1)^s C_{2s}^{n} \tan^{2s}\theta}\,, \quad \text{where } n = 2k+1\ .}$$

e.g. $n=2, k=1:$ $\quad \tan 2\theta = \frac{\sum_{s=0}^{0} (-1)^s C_{2s+1}^{2} \tan^{2s+1}\theta}{\sum_{s=0}^{1} (-1)^s C_{2s}^{2} \tan^{2s}\theta} = \frac{2\tan\theta}{1-\tan^2\theta}$

$n=4, k=2:$ $\quad \tan 4\theta = \frac{\sum_{s=0}^{1} (-1)^s C_{2s+1}^{4} \tan^{2s+1}\theta}{\sum_{s=0}^{2} (-1)^s C_{2s}^{4} \tan^{2s}\theta} = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

An Interesting Example: Equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ can be expressed as:

$x^4 - 6x^2 + 1 = 4x - 4x^3\,, \qquad \text{i.e. } 1 = \frac{4x - 4x^3}{1 - 6x^2 + x^4}$

Let $x = \tan\theta$, then $RHS = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = \tan 4\theta$

$\therefore \quad \tan 4\theta = 1\,, \qquad 4\theta = k\pi + \frac{\pi}{4}$ where $k = 0, 1, 2, 3$

$\theta = (4k+1)\,\frac{\pi}{16} = \frac{\pi}{16},\ \frac{5\pi}{16},\ \frac{9\pi}{16},\ \frac{13\pi}{16}$

$x = \tan^{-1}\left(\frac{\pi}{16}\right),\ \tan^{-1}\left(\frac{5\pi}{16}\right),\ \tan^{-1}\left(\frac{9\pi}{16}\right),\ \tan^{-1}\left(\frac{13\pi}{16}\right)$