Trigonimetrical Multi-angle Expansion using Complex Numbers – The Working Out

In this context, $z = \cos\theta + i\sin\theta$, $\quad \therefore z^n = \cos n\theta + i\sin n\theta$, where $n \in \mathbb{R}+$

$$z^n = (\cos\theta + i\sin\theta)^n = \sum_{r=0}^{n} C_r^n \cos^{n-r}\theta \cdot i^r \sin^r\theta$$

When $n = 2k+1$ where k is an integer (i.e. n is odd), we start by separating the odd and even terms:

$$z^n = \sum_{s=0}^{k} C_{2s}^n \cos^{n-2s}\theta \cdot i^{2s}\sin^{2s}\theta + \sum_{s=0}^{k} C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot i^{2s+1}\sin^{2s+1}\theta \quad \text{(Note: } r = 2s\text{; when } s = k, 2s+1 = n.)$$

$$= \sum_{s=0}^{k} C_{2s}^n \cos^{n-2s}\theta \cdot (i^2)^s \sin^{2s}\theta + \sum_{s=0}^{k} C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot (i^2)^s \cdot i\sin^{2s+1}\theta$$

$$= \sum_{s=0}^{k} (-1)^s C_{2s}^n \cos^{n-2s}\theta \cdot \sin^{2s}\theta + i \cdot \sum_{s=0}^{k} (-1)^s C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta$$

$$= \cos n\theta + i\sin n\theta$$

$$\therefore \boxed{\cos n\theta = \sum_{s=0}^{k} (-1)^s C_{2s}^n \cos^{n-2s}\theta \cdot \sin^{2s}\theta\,, \quad \text{where } n = 2k+1\,.}$$

e.g. $n = 3, k = 1$: $\quad \cos 3\theta = \sum_{s=0}^{1} (-1)^s C_{2s}^3 \cos^{3-2s}\theta \cdot \sin^{2s}\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta$$

$n = 5, k = 2$: $\quad \cos 5\theta = \sum_{s=0}^{2} (-1)^s C_{2s}^5 \cos^{5-2s}\theta \cdot \sin^{2s}\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

$$\boxed{\sin n\theta = \sum_{s=0}^{k} (-1)^s C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta\,, \quad \text{where } n = 2k+1\,.}$$

e.g. $n = 3, k = 1$: $\quad \sin 3\theta = \sum_{s=0}^{1} (-1)^s C_{2s+1}^3 \cos^{3-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

$n = 5, k = 2$: $\quad \sin 5\theta = \sum_{s=0}^{2} (-1)^s C_{2s+1}^5 \cos^{5-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 5\cos^4\theta \cdot \sin\theta - 10\cos^2\theta \cdot \sin^3\theta + \sin^5\theta$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{\sum_{s=0}^{k}(-1)^s C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta}{\sum_{s=0}^{k}(-1)^s C_{2s}^n \cos^{n-2s}\theta \cdot \sin^{2s}\theta}$$

$$= \frac{\cos^n\theta \cdot \sum_{s=0}^{k}(-1)^s C_{2s+1}^n \cos^{-(2s+1)}\theta \cdot \sin^{2s+1}\theta}{\cos^n\theta \cdot \sum_{s=0}^{k}(-1)^s C_{2s}^n \cos^{-2s}\theta \cdot \sin^{2s}\theta}$$

$$\therefore \boxed{\tan n\theta = \frac{\sum_{s=0}^{k}(-1)^s C_{2s+1}^n \tan^{2s+1}\theta}{\sum_{s=0}^{k}(-1)^s C_{2s}^n \tan^{2s}\theta}\,, \quad \text{where } n = 2k+1\,.}$$

e.g. $n = 3, k = 1$: $\quad \tan 3\theta = \dfrac{\sum_{s=0}^{1}(-1)^s C_{2s+1}^3 \tan^{2s+1}\theta}{\sum_{s=0}^{1}(-1)^s C_{2s}^3 \tan^{2s}\theta} = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$n = 5, k = 2$: $\quad \tan 5\theta = \dfrac{\sum_{s=0}^{2}(-1)^s C_{2s+1}^5 \tan^{2s+1}\theta}{\sum_{s=0}^{2}(-1)^s C_{2s}^5 \tan^{2s}\theta} = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

When $n = 2k$ where k is an integer (i.e. n is even), we start by separating the odd and even terms:

$$\begin{aligned}
z^n &= \sum_{s=0}^{k} C_{2s}^n \cos^{n-2s}\theta \cdot i^{2s} \sin^{2s}\theta + \sum_{s=0}^{k-1} C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot i^{2s+1} \sin^{2s+1}\theta \quad \text{(Note: } r = 2s\text{; when } s = k, 2s = n.) \\
&= \sum_{s=0}^{k} C_{2s}^n \cos^{n-2s}\theta \cdot (i^2)^s \sin^{2s}\theta + \sum_{s=0}^{k-1} C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot (i^2)^s \cdot i \sin^{2s+1}\theta \\
&= \sum_{s=0}^{k} (-1)^s C_{2s}^n \cos^{n-2s}\theta \cdot \sin^{2s}\theta + i \cdot \sum_{s=0}^{k-1} (-1)^s C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta \\
&= \cos n\theta + i \sin n\theta
\end{aligned}$$

$$\therefore \quad \boxed{\cos n\theta = \sum_{s=0}^{k} (-1)^s C_{2s}^n \cos^{n-2s}\theta \cdot \sin^{2s}\theta \,, \quad \text{where } n = 2k \,.}$$

$$\begin{aligned}
\text{e.g. } n = 2, k = 1: &\quad \cos 2\theta = \sum_{s=0}^{1} (-1)^s C_{2s}^2 \cos^{2-2s}\theta \cdot \sin^{2s}\theta = \cos^2\theta - \sin^2\theta \\
n = 4, k = 2: &\quad \cos 4\theta = \sum_{s=0}^{2} (-1)^s C_{2s}^4 \cos^{4-2s}\theta \cdot \sin^{2s}\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta
\end{aligned}$$

$$\boxed{\sin n\theta = \sum_{s=0}^{k-1} (-1)^s C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta \,, \quad \text{where } n = 2k \,.}$$

$$\begin{aligned}
\text{e.g. } n = 2, k = 1: &\quad \sin 2\theta = \sum_{s=0}^{0} (-1)^s C_{2s+1}^2 \cos^{2-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 2\cos\theta \sin\theta \\
n = 4, k = 2: &\quad \sin 4\theta = \sum_{s=0}^{1} (-1)^s C_{2s+1}^4 \cos^{4-(2s+1)}\theta \cdot \sin^{2s+1}\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta \\
&\quad = 4\cos\theta \sin\theta \,(\cos^2\theta - \sin^2\theta) \qquad (= 2 \cdot \sin 2\theta \cdot \cos 2\theta)
\end{aligned}$$

$$\begin{aligned}
\tan n\theta = \frac{\sin n\theta}{\cos n\theta} &= \frac{\sum_{s=0}^{k-1} (-1)^s C_{2s+1}^n \cos^{n-(2s+1)}\theta \cdot \sin^{2s+1}\theta}{\sum_{s=0}^{k} (-1)^s C_{2s}^n \cos^{n-2s}\theta \cdot \sin^{2s}\theta} \\
&= \frac{\cos^n\theta \cdot \sum_{s=0}^{k-1} (-1)^s C_{2s+1}^n \cos^{-(2s+1)}\theta \cdot \sin^{2s+1}\theta}{\cos^n\theta \cdot \sum_{s=0}^{k} (-1)^s C_{2s}^n \cos^{-2s}\theta \cdot \sin^{2s}\theta}
\end{aligned}$$

$$\therefore \quad \boxed{\tan n\theta = \frac{\sum_{s=0}^{k-1} (-1)^s C_{2s+1}^n \tan^{2s+1}\theta}{\sum_{s=0}^{k} (-1)^s C_{2s}^n \tan^{2s}\theta} \,, \quad \text{where } n = 2k + 1 \,.}$$

$$\begin{aligned}
\text{e.g. } n = 2, k = 1: &\quad \tan 2\theta = \frac{\sum_{s=0}^{0} (-1)^s C_{2s+1}^2 \tan^{2s+1}\theta}{\sum_{s=0}^{1} (-1)^s C_{2s}^2 \tan^{2s}\theta} = \frac{2\tan\theta}{1 - \tan^2\theta} \\
n = 4, k = 2: &\quad \tan 4\theta = \frac{\sum_{s=0}^{1} (-1)^s C_{2s+1}^4 \tan^{2s+1}\theta}{\sum_{s=0}^{2} (-1)^s C_{2s}^4 \tan^{2s}\theta} = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}
\end{aligned}$$