Trigonimetrical Function Expansion using Complex Numbers

Useful in integration by expanding $\sin^n\theta$ or $\cos^n\theta$ into a series of $\sin k\theta$ or $\cos k\theta$.

In this context, $z = \cos\theta + i\sin\theta$, $\quad \therefore |z| = |z^n| = |z^{-1}| = |z^{-n}| = 1$, where $n \in \mathbb{R}+$

$$z^n = \cos n\theta + i\sin n\theta \,, \qquad z^{-n} = \cos n\theta - i\sin n\theta$$

$$z^n + z^{-n} = 2\cos n\theta \,, \qquad z^n - z^{-n} = 2i\sin n\theta$$

$$(z+z^{-1})^n = 2^n\cos^n\theta \,, \qquad (z-z^{-1})^n = 2^n i^n \sin^n\theta$$

$$\boxed{\cos^n\theta = \frac{1}{2^{n-1}}\sum_{r=0}^{k} C_r^n \cdot \cos(n-2r)\theta \,, \quad \text{where } n = 2k+1 \,.}$$

e.g. $n=3, k=1:$ $\quad \cos^3\theta = \frac{1}{2^{3-1}}\sum_{r=0}^{1} C_r^3 \cdot \cos(3-2r)\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$, $\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$n=5, k=2:$ $\quad \cos^3\theta = \frac{1}{2^{5-1}}\sum_{r=0}^{2} C_r^5 \cdot \cos(5-2r)\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

$$\boxed{\cos^n\theta = \frac{1}{2^{n-1}}\left(\frac{1}{2}C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot \cos(n-2r)\theta\right) \,, \quad \text{where } n = 2k \,.}$$

e.g. $n=2, k=1:$ $\quad \cos^2\theta = \frac{1}{2^{2-1}}\left(\frac{1}{2}C_1^2 + \sum_{r=0}^{0} C_r^2 \cdot \cos(2-2r)\theta\right) = \frac{1}{2}(1+\cos 2\theta)$, $\quad \cos 2\theta = 2\cos^2\theta - 1$

$n=4, k=2:$ $\quad \cos^4\theta = \frac{1}{2^{4-1}}\left(\frac{1}{2}C_2^4 + \sum_{r=0}^{1} C_r^4 \cdot \cos(4-2r)\theta\right) = \frac{1}{8}(3+\cos 4\theta + 4\cos 2\theta)$

$$\boxed{\sin^n\theta = \frac{(-1)^k}{2^{n-1}}\sum_{r=0}^{k}(-1)^r\, C_r^n \cdot \sin(n-2r)\theta \,, \quad \text{where } n = 2k+1 \,.}$$

e.g. $n=3, k=1:$ $\quad \sin^3\theta = \frac{(-1)^1}{2^{3-1}}\sum_{r=0}^{1}(-1)^r\, C_r^3 \cdot \sin(3-2r)\theta = \frac{-1}{4}(\sin 3\theta - 3\sin\theta)$, $\quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$n=5, k=2:$ $\quad \sin^5\theta = \frac{(-1)^2}{2^{5-1}}\sum_{r=0}^{2}(-1)^r\, C_r^5 \cdot \sin(5-2r)\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

$$\boxed{\sin^n\theta = \frac{1}{2^{n-1}}\left(\frac{1}{2}C_k^n + (-1)^k\sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot \cos(n-2r)\theta\right) \,, \quad \text{where } n = 2k+1 \,.}$$

e.g. $n=2, k=1:$ $\quad \sin^2\theta = \frac{1}{2^{2-1}}\left(\frac{1}{2}C_1^2 + (-1)^1\sum_{r=0}^{0}(-1)^r\, C_r^2 \cdot \cos(2-2r)\theta\right) = \frac{1}{2}(1-\cos 2\theta)$, $\quad \cos 2\theta = 1 - 2\sin^2\theta$

$n=4, k=2:$ $\quad \sin^4\theta = \frac{1}{2^{4-1}}\left(\frac{1}{2}C_2^4 + (-1)^2\sum_{r=0}^{1}(-1)^r\, C_r^4 \cdot \cos(4-2r)\theta\right) = \frac{1}{8}(3+\cos 4\theta - 4\cos 2\theta)$

Ways to remember:

1. Coefficents of terms inside the brackets are taken from the first half of the Pascal's Triangle. (1, 1-3, 1-4, 1-5-10, 1-6-15, 1-7-21-35, 1-8-28-56, ...)
2. For cos, all coefficients are positive; for sin, the first term is positive if n or $n-1$ is divisible by 4, or negative otherwise. The signs of the rest of the terms alternate.
3. Coefficient of θ starts from n and decreases by 2 each term on. (2, 3-1, 4-2, 5-3-1, 6-4-2, 7-5-3-1, 8-6-4-2, ...)
4. When n is even (i.e. odd Pascal's numbers), divide the middle number by 2 and add to the sequence. (2÷2, 6÷2, 20÷2, 70÷2, ...)
5. The expansion of $\sin^n\theta$ with an even n contains a series of $\cos k\theta$ (instead of $\sin k\theta$ as one would expect).