Trigonimetrical Function Expansion using Complex Numbers – The Working Out

cos Expansions: $2^n \cos^n \theta = (z + z^{-1})^n = \sum_{r=0}^{n} C_r^n \cdot (z)^{n-r} \cdot (z^{-1})^r = \sum_{r=0}^{n} C_r^n \cdot z^{n-2r}$

When $n = 2k+1$ where k is an integer (i.e. n is odd),

$$\begin{aligned}
2^n \cos^n \theta &= \sum_{r=0}^{k} C_r^n \cdot z^{n-2r} + \sum_{r=k+1}^{n} C_r^n \cdot z^{n-2r} \\
&= \sum_{r=0}^{k} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{n-(k+1)} C_{r+(k+1)}^n \cdot z^{n-2[r+(k+1)]} \\
&= \sum_{r=0}^{k} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} C_{(k-r)+(k+1)}^n \cdot z^{n-2[(k-r)+(k+1)]} \\
&= \sum_{r=0}^{k} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} C_{(2k+1)-r}^n \cdot z^{n-2[(2k+1)-r]} \\
&= \sum_{r=0}^{k} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} C_{n-r}^n \cdot z^{n-2[n-r]} \\
&= \sum_{r=0}^{k} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} C_r^n \cdot z^{-(n-2r)} \\
&= \sum_{r=0}^{k} C_r^n \cdot \left(z^{(n-2r)} + z^{-(n-2r)} \right) \\
&= \sum_{r=0}^{k} C_r^n \cdot 2\cos(n-2r)\theta \\
\therefore 2^n \cos^n \theta &= \sum_{r=0}^{k} C_r^n \cdot 2\cos(n-2r)\theta
\end{aligned}$$

$$\boxed{\cos^n \theta = \frac{1}{2^{n-1}} \sum_{r=0}^{k} C_r^n \cdot \cos(n-2r)\theta\,, \quad \text{where } n = 2k+1\ .}$$

e.g. $n = 3, k = 1:\quad \cos^3 \theta = \frac{1}{2^{3-1}} \sum_{r=0}^{1} C_r^3 \cdot \cos(3-2r)\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)\ , \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$n = 5, k = 2:\quad \cos^5 \theta = \frac{1}{2^{5-1}} \sum_{r=0}^{2} C_r^5 \cdot \cos(5-2r)\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

$n = 7, k = 3:\quad \cos^7 \theta = \frac{1}{2^{7-1}} \sum_{r=0}^{3} C_r^7 \cdot \cos(7-2r)\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21 \cdot \cos 3\theta + 35\cos\theta)$

When $n = 2k$ where k is an integer (i.e. n is even),

$$
\begin{aligned}
2^n \cos^n \theta &= \sum_{r=0}^{k-1} C_r^n \cdot z^{n-2r} + C_k^n \cdot z^{n-2k} + \sum_{r=k+1}^{n} C_r^n \cdot z^{n-2r} \\
&= \sum_{r=0}^{k-1} C_r^n \cdot z^{n-2r} + C_k^n \cdot z^0 + \sum_{r=0}^{n-(k+1)} C_{r+(k+1)}^n \cdot z^{n-2[r+(k+1)]} \\
&= \sum_{r=0}^{k-1} C_r^n \cdot z^{n-2r} + C_k^n + \sum_{r=0}^{k-1} C_{(k-1-r)+(k+1)}^n \cdot z^{n-2[(k-1-r)+(k+1)]} \\
&= C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k-1} C_{2k-r}^n \cdot z^{n-2[2k-r]} \\
&= C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k-1} C_{n-r}^n \cdot z^{n-2[n-r]} \\
&= C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k-1} C_r^n \cdot z^{-(n-2r)} \\
&= C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot \left(z^{(n-2r)} + z^{-(n-2r)} \right) \\
&= C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot 2\cos(n-2r)\theta \\
\therefore 2^n \cos^n \theta &= C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot 2\cos(n-2r)\theta
\end{aligned}
$$

$$
\boxed{\cos^n \theta = \frac{1}{2^{n-1}} \left(\frac{1}{2} C_k^n + \sum_{r=0}^{k-1} C_r^n \cdot \cos(n-2r)\theta \right) , \quad \text{where } n = 2k \ .}
$$

$$
\begin{aligned}
\text{e.g. } n = 2, k = 1 : \quad & \cos^2 \theta = \frac{1}{2^{2-1}} \left(\frac{1}{2} C_1^2 + \sum_{r=0}^{0} C_r^2 \cdot \cos(2-2r)\theta \right) = \frac{1}{2} \left(1 + \cos 2\theta \right) \ , \quad \cos 2\theta = 2\cos^2\theta - 1 \\
n = 4, k = 2 : \quad & \cos^4 \theta = \frac{1}{2^{4-1}} \left(\frac{1}{2} C_2^4 + \sum_{r=0}^{1} C_r^4 \cdot \cos(4-2r)\theta \right) = \frac{1}{8} \left(3 + \cos 4\theta + 4\cos 2\theta \right) \\
n = 6, k = 3 : \quad & \cos^6 \theta = \frac{1}{2^{6-1}} \left(\frac{1}{2} C_3^6 + \sum_{r=0}^{2} C_r^6 \cdot \cos(6-2r)\theta \right) = \frac{1}{32} \left(10 + \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta \right)
\end{aligned}
$$

sin Expansions: $2^n i^n \sin^n\theta = (z - z^{-1})^n = \sum_{r=0}^{n} (-1)^r\, C_r^n \cdot (z)^{n-r} \cdot (z^{-1})^r = \sum_{r=0}^{n} (-1)^r\, C_r^n \cdot z^{n-2r}$

When $n = 2k+1$ where k is an integer (i.e. n is odd),

$$2^n i^n \sin^n\theta = \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=k+1}^{n} (-1)^r\, C_r^n \cdot z^{n-2r}$$

$$2^n\, i^{2k+1} \sin^n\theta = \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{n-(k+1)} (-1)^{r+(k+1)}\, C_{r+(k+1)}^n \cdot z^{n-2[r+(k+1)]}$$

$$2^n (i^2)^k\, i \sin^n\theta = \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} (-1)^{(k-r)+(k+1)}\, C_{(k-r)+(k+1)}^n \cdot z^{n-2[(k-r)+(k+1)]}$$

$$2^n (-1)^k\, i \sin^n\theta = \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} (-1)^{(2k+1)-r}\, C_{(2k+1)-r}^n \cdot z^{n-2[(2k+1)-r]}$$

$$= \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} (-1)^{n-r}\, C_{n-r}^n \cdot z^{n-2[n-r]}$$

$$= \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k} (-1)^{n-r}\, C_r^n \cdot z^{-(n-2r)}$$

$$= \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot \left(z^{(n-2r)} + (-1)^{n-2r}\, z^{-(n-2r)} \right)$$

$$= \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot \left(z^{(n-2r)} - z^{-(n-2r)} \right)$$

$$= \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot 2i \sin(n-2r)\theta$$

$$\therefore\ 2^n \sin^n\theta = (-1)^k \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot 2\sin(n-2r)\theta$$

$$\boxed{\sin^n\theta = \frac{(-1)^k}{2^{n-1}} \sum_{r=0}^{k} (-1)^r\, C_r^n \cdot \sin(n-2r)\theta\ , \quad \text{where } n = 2k+1\ .}$$

e.g. $n=3, k=1:\quad \sin^3\theta = \frac{(-1)^1}{2^{3-1}} \sum_{r=0}^{1} (-1)^r\, C_r^3 \cdot \sin(3-2r)\theta = \frac{1}{4}\left(-\sin 3\theta + 3\sin\theta\right)\ , \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$n=5, k=2:\quad \sin^5\theta = \frac{(-1)^2}{2^{5-1}} \sum_{r=0}^{2} (-1)^r\, C_r^5 \cdot \sin(5-2r)\theta = \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)$

$n=7, k=3:\quad \sin^7\theta = \frac{(-1)^3}{2^6} \sum_{r=0}^{3} (-1)^r\, C_r^7 \cdot \sin(7-2r)\theta = \frac{1}{64}\left(-\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta\right)$

When $n = 2k$ where k is an integer (i.e. n is even),

$$2^n i^n \sin^n\theta = \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot z^{n-2r} + (-1)^k\, C_k^n \cdot z^{n-2k} + \sum_{r=k+1}^{n}(-1)^r\, C_r^n \cdot z^{n-2r}$$

$$2^n\, i^{2k}\sin^n\theta = \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot z^{n-2r} + (-1)^k\, C_k^n \cdot z^0 + \sum_{r=0}^{n-(k+1)}(-1)^{r+(k+1)}\, C_{r+(k+1)}^n \cdot z^{n-2[r+(k+1)]}$$

$$2^n (i^2)^k\sin^n\theta = \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot z^{n-2r} + (-1)^k\, C_k^n + \sum_{r=0}^{k-1}(-1)^{(k-1-r)+(k+1)}\, C_{(k-1-r)+(k+1)}^n \cdot z^{n-2[(k-1-r)+(k+1)]}$$

$$2^n (-1)^k\sin^n\theta = (-1)^{k-1}\, C_{k-1}^n + \sum_{r=0}^{k}(-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k-1}(-1)^{2k-r}\, C_{2k-r}^n \cdot z^{n-2[2k-r]}$$

$$= (-1)^k\, C_k^n + \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k-1}(-1)^{n-r}\, C_{n-r}^n \cdot z^{n-2[n-r]}$$

$$= (-1)^k\, C_k^n + \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot z^{n-2r} + \sum_{r=0}^{k-1}(-1)^{n-r}\, C_r^n \cdot z^{-(n-2r)}$$

$$= (-1)^k\, C_k^n + \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot \left(z^{(n-2r)} + (-1)^{n-2r}\, z^{-(n-2r)}\right)$$

$$= (-1)^k\, C_k^n + \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot \left(z^{(n-2r)} + z^{-(n-2r)}\right)$$

$$= (-1)^k\, C_k^n + \sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot 2\cos(n-2r)\theta$$

$$\therefore 2^n\sin^n\theta = C_k^n + (-1)^k\sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot 2\cos(n-2r)\theta$$

$$\boxed{\sin^n\theta = \frac{1}{2^{n-1}}\left(\frac{1}{2}C_k^n + (-1)^k\sum_{r=0}^{k-1}(-1)^r\, C_r^n \cdot \cos(n-2r)\theta\right)\ , \quad \text{where } n = 2k+1\ .}$$

e.g. $n=2, k=1:\quad \sin^2\theta = \frac{1}{2^{2-1}}\left(\frac{1}{2}C_1^2 + (-1)^1\sum_{r=0}^{0}(-1)^r\, C_r^2 \cdot \cos(2-2r)\theta\right) = \frac{1}{2}\left(1-\cos 2\theta\right)\ ,\quad \cos 2\theta = 1 - 2\sin^2\theta$

$n=4, k=2:\quad \sin^4\theta = \frac{1}{2^{4-1}}\left(\frac{1}{2}C_2^4 + (-1)^2\sum_{r=0}^{1}(-1)^r\, C_r^4 \cdot \cos(4-2r)\theta\right) = \frac{1}{8}\left(3+\cos 4\theta - 4\cos 2\theta\right)$

$n=6, k=3:\quad \sin^6\theta = \frac{1}{2^{6-1}}\left(\frac{1}{2}C_3^6 + (-1)^3\sum_{r=0}^{2}(-1)^r\, C_r^6 \cdot \cos(6-2r)\theta\right) = \frac{1}{32}\left(10-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta\right)$