Complex Number – Polar Form and De Moivre's Theorem

Modulus: $|\bar{z}| = |-z| = |z|$

$|kz| = k|z|$, where $k \in \mathbb{R}+$.

$z\bar{z} = |z|^2$

$\dfrac{z}{\bar{z}} = \cos 2\theta + i \sin 2\theta$

$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$ $\quad \big((z_1+z_2)(\overline{z_1}+\overline{z_2}) + (z_1-z_2)(\overline{z_1}-\overline{z_2}) = 2\, z_1\, \overline{z_1} + 2\, z_2\, \overline{z_2}\big)$

$|z_1 + z_2| \leq |z_1| + |z_2|$

$\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ $\quad \big(|z_1| = |(z_1+z_2) + (-z_2)| \leq |z_1+z_2| + |-z_2|,\ |z_1| - |z_2| \leq |z_1+z_2|\big)$

Given $z_1 \neq 0$ and $z_2 \neq 0$, $|z_1 + z_2| = |z_1| + |z_2| \;\Leftrightarrow\; \dfrac{z_1}{z_2} \in \mathbb{R} \;\Leftrightarrow\; O$, P_1 and P_2 are collinear.

Arguments: Principal argument value is in either $(-\pi, \pi]$ (to match the range of $\tan^{-1}$) or $[0, 2\pi)$ (to be positive).

Here we denote the principal argument value of complex number z with $arg(z) \in (-\pi, \pi]$.

If a formula may result in an argument value out of the $(-\pi, \pi]$ domain, mod 2π is required.

$\arg(z) = 0$, If z is purely real and $z \geq 0$.

$\arg(z) = \pi$, If z is purely real and $z < 0$.

$\arg(z) = \frac{\pi}{2}$, If z is purely imaginary and $z = ki$ where $k \in \mathbb{R}+$.

$\arg(z) = -\frac{\pi}{2}$, If z is purely imaginary and $z = ki$ where $k \in \mathbb{R}-$.

$\arg(-z) = \arg(z) - \pi$, where $\arg(z) \neq 0$

$\arg(\bar{z}) = -\arg(z)$, where $\arg(z) \neq \pi$

$\arg(z_1 \cdot z_2) \equiv arg(z_1) + arg(z_2) \pmod{2\pi}$ $\quad \Big($e.g. $\arg(\text{cis}(\pi) \cdot \text{cis}(\frac{\pi}{2})) \equiv \frac{3\pi}{2} \equiv -\frac{1}{\pi} \pmod{2\pi}\Big)$

$\arg(z^n) \equiv n\ \arg(z) \pmod{2\pi}$

$\arg\left(\dfrac{1}{z}\right) = -\arg(z)$, where $\arg(z) \neq \pi$

$\arg\left(\dfrac{z_1}{z_2}\right) \equiv \arg(z_1) - \arg(z_2) \pmod{2\pi}$

De Moivre's Theorem: $[r_1(\cos\theta_1 + i\sin\theta_1)] \cdot [r_2(\cos\theta_2 + i\sin\theta_2)] = (r_1 \cdot r_2)[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Proof: $LHS = (r_1 \cdot r_2)[(\cos\theta_1\ \cos\theta_2 - \sin\theta_1\ \sin\theta_2) + i(\sin\theta_1\ \cos\theta_2 + \cos\theta_1\ \sin\theta_2)]$

$= (r_1 \cdot r_2)[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] = RHS$

$\overline{(\cos\theta + i\sin\theta)} = \cos(-\theta) + i\sin(-\theta)$

$\frac{1}{\cos\theta + i\sin\theta} = \cos(-\theta) + i\sin(-\theta)$

$(\cos\theta + i\sin\theta)^0 = \cos 0 + i\sin 0 = 1$

If n and m are positive integers,

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $\quad (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$(\cos\theta + i\sin\theta)^{\frac{n}{m}} = \cos\frac{n}{m}\theta + i\sin\frac{n}{m}\theta$, $\quad (\cos\theta + i\sin\theta)^{-\frac{n}{m}} = \cos(-\frac{n}{m}\theta) + i\sin(-\frac{n}{m}\theta)$