Complex Exponential

Definition: $\boxed{e^{ix} = \cos x + i \sin x\,,}$ This is equivalent to the shorthand of cis(x).

It is known that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \text{and}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,.$$

So $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots = \left[1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \ldots\right] + \left[(ix) + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \ldots\right]$

$$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right] = \cos x + i \sin x\,.$$

Generally, $|e^{ix}| = 1$ and $\arg(e^{ix}) = x$.

$$|e^z| = \left|e^{\mathrm{Re}(z)+i\mathrm{Im}(z)}\right| = \left|e^{\mathrm{Re}(z)} \cdot e^{i\mathrm{Im}(z)}\right| = \left|e^{\mathrm{Re}(z)}\right| \cdot \left|e^{i\mathrm{Im}(z)}\right| = e^{\mathrm{Re}(z)}\,.$$

$$\arg(e^z) = \arg\left(e^{\mathrm{Re}(z)+i\mathrm{Im}(z)}\right) = \arg\left(e^{\mathrm{Re}(z)} \cdot e^{i\mathrm{Im}(z)}\right) = \arg\left(e^{i\mathrm{Im}(z)}\right) = \mathrm{Im}(z)\,.$$

Logarithm: $\boxed{\ln(z) = \ln|z| + i \arg(z)\,.}$

Let $z = r(\cos\theta + i\,\sin\theta)$ and $\ln(z) = a + ib$.

$z = e^{a+ib} = e^a \cdot e^{ib} = e^a \cdot (\cos b + i \sin b)$. So $e^a \cos b = r\cos\theta \ldots (1)$ and $e^a \sin b = r \sin\theta \ldots (2)$.

$(2) \div (1)$: $\tan b = \tan\theta$. $\therefore b = \theta$. (Only the smallest positive solution is used.)

$(1)^2 + (2)^2$: $e^{2a}(\cos^2 b + \sin^2 b) = r^2(\cos^2\theta + \sin^2\theta)$, $e^a = r\ (> 0)$. $\therefore a = \ln(r)$.

$ln(z) = a + ib = \ln(r) + i\,\theta$.

Revisit: The exponential form is simpler and more natural.

Multiplication:

$$r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i \sin\theta_2) = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]\,.$$

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1 + i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}\,.$$

de Moivre's Theorem:

$$[r(\cos\theta + i\sin\theta)]^k = r(\cos k\theta + i \sin k\theta)\,, \quad \text{where } k \in \mathbb{R}\,.$$

$$(re^{i\theta})^k = r^k e^{ik\theta}\,.$$

Differentiation:

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta)\,.$$

$$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}.$$

Integration:

$$\int (\cos\theta + i\sin\theta)d\theta = \sin\theta - i\cos\theta + C = -i(\cos\theta + i\sin\theta) + C = \frac{1}{i}(\cos\theta + i\sin\theta) + C\,.$$

$$\int e^{i\theta} d\theta = \frac{1}{i}e^{i\theta} + C\,.$$