Complex Numbers

Definition: $i \equiv \sqrt{-1}$ where i is an *imaginary* number that satisfies $i^2 = -1$.

(If there is a $j \neq i$ and $j^2 = -1$ as well, then it must be that $j = -i = -\sqrt{-1}$, the *conjugate* of i.)

Cartesian: $z = a + ib$

Polar: $z = r\,(\cos\theta + i\sin\theta) = r \text{ cis}\,\theta = r\,e^{i\theta}$, where $r > 0$ (if $z \neq 0$) and $0 \leq \theta < 2\pi$.

(When $z = 0$, it is not presented in polar form, as $\arg(0)$ does not exist.)

Convertions: $r = |z| \equiv \sqrt{a^2 + b^2}\,, \quad \theta = \arg(z) \equiv \tan^{-1}\dfrac{b}{a}\,, \quad \text{Re}(z) \equiv a = r\cos\theta\,, \quad \text{Im}(z) \equiv b = r\sin\theta$

Basics: Algebra for $\mathbb{R}$ works mostly but not always ... e.g. $\sqrt{a \times b} \neq \sqrt{a} \times \sqrt{b}$ when $a < 0$ and $b < 0$.

Instead $\sqrt{a \times b} = i\sqrt{-a} \times i\sqrt{-b} = i^2\sqrt{-a} \times \sqrt{-b} = -\sqrt{-a} \times \sqrt{-b}$ (Note: $-a > 0$ and $-b > 0$)

$\text{Re}(z_1 \pm z_2) = \text{Re}(z_1) \pm \text{Re}(z_2)\,, \quad \text{Im}(z_1 \pm z_2) = \text{Im}(z_1) \pm \text{Im}(z_2)\,, \quad |\text{Re}(z)| \leq |z|\,, \quad |\text{Im}(z)| \leq |z|$

If $z = x + iy$ and $z^2 = a + ib$ where $a, b, x, y \in \mathbb{R}$, then $x^2 - y^2 = a$ and $2xy = b$. $\left(\sqrt{a + ib} = x + iy\right)$

If $z_1 = z_2$, then $\text{Re}(z_1)=\text{Re}(z_2)$ and $\text{Im}(z_1)=\text{Im}(z_2)$.

Conjugate: $\bar{z} \equiv \text{Re}(z) - i\text{Im}(z) = a - ib$ or $\bar{z} \equiv |z|\left[\cos\big(-\arg(z)\big) + i\sin\big(-\arg(z)\big)\right] = r\left[\cos(-\theta) + i\sin(-\theta)\right]$

For any polynomial equation $\sum_{k=0}^{n} a_k x^k = 0$ where $a_k \in \mathbb{R}$, if α is a complex solution, so is $\overline{\alpha}$.

$\text{Re}(z) = \dfrac{z + \bar{z}}{2}\,, \quad \text{Im}(z) = \dfrac{z - \bar{z}}{2i}\,, \quad \overline{(z_1 \pm z_2)} = \overline{z_1} \pm \overline{z_2}\,, \quad \overline{(z_1 \cdot z_2)} = \overline{z_1} \cdot \overline{z_2}\,, \quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

General Form of Equation: $\boxed{z^n = a + ib}$

Let $z = R \text{ cis}\,\phi\,, \quad a + ib = r \text{ cis}\,\theta$

$(R \text{ cis}\,\phi)^n = r \text{ cis}\,\theta$

$R^n \text{ cis}\,n\phi = r \text{ cis}\,\theta$

$R^n = r\,, \quad n\phi = \theta + 2k\pi$ where $k \in \mathbb{Z}$

$R = r^{\frac{1}{n}}\,, \quad \phi = \dfrac{\theta + 2k\pi}{n} = \dfrac{\theta}{n} + \dfrac{2\pi}{n}k$

Solutions are $\boxed{\sqrt[n]{r} \cdot \left[\cos\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right) + i \cdot \sin\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)\right] \quad \text{where } \theta = \tan^{-1}\tfrac{b}{a},\ k = 0, 1, \ldots, n-1}$

Roots of Unity: $\boxed{z^n = 1}$ General Form with $a = 1$, $b = 0$ and $\theta = 0$

Solutions are $\boxed{\cos\left(\frac{2\pi}{n}k\right) + i \cdot \sin\left(\frac{2\pi}{n}k\right) \quad \text{where } k = 0, 1, \ldots, n-1}$

Let each solution be $w_k = \cos\left(\frac{2\pi}{n}k\right) + i \cdot \sin\left(\frac{2\pi}{n}k\right)$, and $w = w_1$, then these will follow:

$w_k = w^k = w^{k+mn}$ where $m \in \mathbb{Z}$. $(w_0 = w^0 = w^n = 1)$ w^k is used hereinafter, instead of w_k .

All w^k values form a unit circle.

$(w^k)^n = 1$ $\because w^k$ is a solution.

$\sum_{k=0}^{n-1} w^k = 1 + w + w^2 + \ldots + w^{n-1} = \dfrac{w^n - 1}{w - 1} = 0 \quad (w^n = 1, w \neq 1)$

$\overline{w_k} = w^{-k} = w^{n-k}$ i.e. All non-real roots form pairs of conjugates.

$(w - 1) \cdot \sum_{k=0}^{n-1}(k+1)w^k = (w - 1) \cdot (1 + 2w + 3w^2 + \ldots + nw^{n-1}) = n$