Find $\quad I = \displaystyle\int \sqrt{1+\sin x}\, dx$

Method I:

$$\begin{aligned} I &= \int \sqrt{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx \\ &= \int \sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}\,dx \\ &= \int \left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|\,dx \end{aligned}$$

When $x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\quad \frac{x}{2} \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$,

$\sin\frac{x}{2}+\cos\frac{x}{2} \geq 0$, and

$$\begin{aligned} I &= \int \sin\frac{x}{2}+\cos\frac{x}{2}\,dx \\ &= 2\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)+C, \quad \text{where } x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{aligned}$$

Method II:

$$\begin{aligned} I &= \int \sqrt{\frac{1-\sin^2 x}{1-\sin x}}\,dx\,, \quad \text{where } x \neq n\pi \\ &= \int \frac{|\cos x|}{\sqrt{1-\sin x}}\,dx \end{aligned}$$

When $x \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, $\quad \cos x > 0$

$$\therefore I = \int \frac{\cos x}{\sqrt{1-\sin x}}\,dx$$

Let $\quad u = 1-\sin x, \quad du = -\cos x\,dx$

$$\begin{aligned} I &= \int \frac{-1}{\sqrt{u}}\,du \\ I &= -2\sqrt{u}+C \\ &= -2\sqrt{1-\sin x}+C \end{aligned}$$

When $x \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right)$, $\quad \cos x < 0$

$$\begin{aligned} \therefore I &= \int \frac{-\cos x}{\sqrt{1-\sin x}}\,dx \\ &= 2\sqrt{1-\sin x}+C \end{aligned}$$

Let $s(x) = -1$ when $\cos x > 0$, and $s(x) = 1$ when $\cos x < 0$, then $\quad s(x) = \dfrac{-\cos x}{\sqrt{\cos^2 x}} \quad$ for $x \neq n\pi + \dfrac{\pi}{2}$

$$\begin{aligned} I &= 2\cdot s(x)\sqrt{1-\sin x}+C \\ &= 2\cdot\frac{-\cos x}{\sqrt{\cos^2 x}}\sqrt{1-\sin x}+C = -2\cos x\sqrt{\frac{1-\sin x}{\cos^2 x}} = -2\cos x\sqrt{\frac{1-\sin x}{1-\sin^2 x}} = \frac{-2\cos x}{\sqrt{1+\sin x}} \end{aligned}$$

$\therefore I = \dfrac{-2\cos x}{\sqrt{1+\sin x}} \quad$ for $x \neq \dfrac{n\pi}{2} \qquad$ (In this question, all restrictions on x fall on $\frac{n\pi}{2}$.)