$$
\begin{aligned}
\text{Find } I &= \int \frac{\cos^4 x}{\sin^3 x}\, dx \\
&= \int \frac{\cos^4 x}{\sin^4 x} \sin x\, dx \\
&= \int \frac{\cos^4 x}{(1-\cos^2 x)^2} \sin x\, dx
\end{aligned}
$$

Let $u = \cos x$, then $du = -\sin x\, dx$

$$
\begin{aligned}
I &= -\int \frac{u^4}{(1-u^2)^2}\, du \\
&= -\int \left(\frac{u^2}{1-u^2}\right)^2 du \\
&= -\int \left[\frac{1-(1-u^2)}{1-u^2}\right]^2 du \\
&= -\int \left(\frac{1}{1-u^2} - 1\right)^2 du \\
&= -\int \frac{1}{(1-u^2)^2} - \frac{2}{1-u^2} + 1\, du \\
&= -\int \left[\frac{2+u}{4(1+u)^2} + \frac{2-u}{4(1-u)^2}\right] - \left[\frac{1}{1+u} + \frac{1}{1-u}\right] + 1\, du \qquad \text{(See note 1)} \\
&= -\int \frac{1}{4}\left[\frac{1+(1+u)}{(1+u)^2} + \frac{1+(1-u)}{(1-u)^2}\right] - \left[\frac{1}{1+u} + \frac{1}{1-u}\right] + 1\, du \\
&= -\int \frac{1}{4}\left[\frac{1}{(1+u)^2} + \frac{1}{1+u} + \frac{1}{(1-u)^2} + \frac{1}{1-u}\right] - \left[\frac{1}{1+u} + \frac{1}{1-u}\right] + 1\, du \\
&= -\int \frac{1}{4}\left[\frac{1}{(1+u)^2} + \frac{1}{(1-u)^2}\right] + \frac{1}{4}\left[\frac{1}{1+u} + \frac{1}{1-u}\right] - \left[\frac{1}{1+u} + \frac{1}{1-u}\right] + 1\, du \\
&= -\int \frac{1}{4}\left[\frac{1}{(1+u)^2} + \frac{1}{(1-u)^2}\right] - \frac{3}{4}\left[\frac{1}{1+u} + \frac{1}{1-u}\right] + 1\, du \\
&= \int -\frac{1}{4}\cdot\frac{1}{(1+u)^2} - \frac{1}{4}\cdot\frac{1}{(1-u)^2} + \frac{3}{4}\cdot\frac{1}{1+u} + \frac{3}{4}\cdot\frac{1}{1-u} - 1\, du \\
&= -\frac{1}{4}\cdot\frac{-2}{1+u} - \frac{1}{4}\cdot\frac{2}{1-u} + \frac{3}{4}\ln(1+u) + \frac{-3}{4}\ln(1-u) - u + C \\
&= \frac{1}{2}\cdot\frac{1}{1+u} - \frac{1}{2}\cdot\frac{1}{1-u} + \frac{3}{4}\ln(1+u) - \frac{3}{4}\ln(1-u) - u + C \\
&= \frac{-2u}{2(1-u^2)} + \frac{3}{4}\ln\left(\frac{1+u}{1-u}\right) - u + C \\
&= \frac{3}{4}\ln\left(\frac{1+u}{1-u}\right) - \frac{u}{(1-u^2)} - u + C
\end{aligned}
$$

$$
\therefore I = \frac{3}{4}\ln\left(\frac{1+\cos x}{1-\cos x}\right) - \frac{\cos x}{\sin^2 x} - \cos x + C
$$

Note 1: $\dfrac{1}{(1-u^2)^2} = \dfrac{A+Bu}{(1+u)^2} + \dfrac{C+Du}{(1-u)^2} = \dfrac{(A+Bu)(1-u)^2 + (C+Du)(1+u)^2}{(1+u)^2(1-u)^2}$

i.e. $(B+D)u^3 + (A-2B+C+2D)u^2 + (-2A+B+2C+D)u + (A+C) = 1$

$B+D = 0\ldots(1),\ A-2B+C+2D = 0\ldots(2),\ -2A+B+2C+D = 0\ldots(3),\ A+C = 1\ldots(4),$

$(1): B = -D, \quad (4): A = 1-C\ldots(5), \quad (2): (1-C) - 2(-D) + C + 2D = 0,$

$1 + 4D = 0, \quad D = -\frac{1}{4}, \quad B = -D = \frac{1}{4}, \quad (3): -2A - D + 2C + D = 0,\ A = C,$

$(5): A = 1-A, \quad 2A = 1, \quad A = \frac{1}{2}, \quad C = A = \frac{1}{2}$