Vector Calculus Identities

Distributed properties:

$\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$

$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

Product rule:

$\nabla(\psi\,\phi) = \phi\,\nabla\psi + \psi\,\nabla\phi$

Product of a scalar and a vector:

$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A}$

$\nabla \times (\psi\mathbf{A}) = \psi\nabla \times \mathbf{A} + \nabla\psi \times \mathbf{A}$

Vector dot product:

$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

Proof: (Notation Note: $c_{nw} \equiv \dfrac{\partial}{\partial w} c_n$)

$$\begin{aligned}
\text{LHS} &= \nabla(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= \mathbf{i}\frac{\partial}{\partial x}(a_1 b_1 + a_2 b_2 + a_3 b_3) + \mathbf{j}\frac{\partial}{\partial y}(a_1 b_1 + a_2 b_2 + a_3 b_3) + \mathbf{k}\frac{\partial}{\partial z}(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= \mathbf{i}\left(\frac{\partial}{\partial x}a_1 b_1 + \frac{\partial}{\partial x}a_2 b_2 + \frac{\partial}{\partial x}a_3 b_3\right) + \mathbf{j}\left(\frac{\partial}{\partial y}a_1 b_1 + \frac{\partial}{\partial y}a_2 b_2 + \frac{\partial}{\partial y}a_3 b_3\right) + \mathbf{k}\left(\frac{\partial}{\partial z}a_1 b_1 + \frac{\partial}{\partial z}a_2 b_2 + \frac{\partial}{\partial z}a_3 b_3\right) \\
&= \left[\mathbf{i}\,(a_1 b_{1x} + a_2 b_{2x} + a_3 b_{3x}) + \mathbf{j}\,(a_1 b_{1y} + a_2 b_{2y} + a_3 b_{3y}) + \mathbf{k}\,(a_1 b_{1z} + a_2 b_{2z} + a_3 b_{3z})\right] \\
&\quad + \left[\mathbf{i}\,(b_1 a_{1x} + b_2 a_{2x} + b_3 a_{3x}) + \mathbf{j}\,(b_1 a_{1y} + b_2 a_{2y} + b_3 a_{3y}) + \mathbf{k}\,(b_1 a_{1z} + b_2 a_{2z} + b_3 a_{3z})\right]
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A} \cdot \nabla)\mathbf{B} &= \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\mathbf{B} \\
&= \mathbf{i}\,(a_1 b_{1x} + a_2 b_{1y} + a_3 b_{1z}) + \mathbf{j}\,(a_1 b_{2x} + a_2 b_{2y} + a_3 b_{2z}) + \mathbf{k}\,(a_1 b_{3x} + a_2 b_{3y} + a_3 b_{3z})
\end{aligned}$$

$$\begin{aligned}
\mathbf{A} \times (\nabla \times \mathbf{B}) &= \mathbf{A} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{A} \times [\mathbf{i}(b_{3y} - b_{2z}) + \mathbf{j}(b_{1z} - b_{3x}) + \mathbf{k}(b_{2x} - b_{1y})] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_{3y} - b_{2z} & b_{1z} - b_{3x} & b_{2x} - b_{1y} \end{vmatrix} \\
&= \mathbf{i}\,(a_2 b_{2x} - a_2 b_{1y} - a_3 b_{1z} + a_3 b_{3x}) + \mathbf{j}\,(a_3 b_{3y} - a_3 b_{2z} - a_1 b_{2x} + a_1 b_{1y}) + \mathbf{k}\,(a_1 b_{1z} - a_1 b_{3x} - a_2 b_{3y} + a_2 b_{2z})
\end{aligned}$$

$(\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} \times (\nabla \times \mathbf{B}) = \mathbf{i}\,(a_1 b_{1x} + a_2 b_{2x} + a_3 b_{3x}) + \mathbf{j}\,(a_1 b_{1y} + a_2 b_{2y} + a_3 b_{3y}) + \mathbf{k}\,(a_1 b_{1z} + a_2 b_{2z} + a_3 b_{3z})$,

which equals the first bracket of the LHS. Similarly, it can be shown $(\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A})$ equals the second. QED

$\nabla(\mathbf{A} \cdot \mathbf{B}) = \nabla_{\mathbf{A}}(\mathbf{A} \cdot \mathbf{B}) + \nabla_{\mathbf{B}}(\mathbf{A} \cdot \mathbf{B})$

Proof:

$$\begin{aligned}
\text{LHS} &= \left[\mathbf{i}\,(a_1 b_{1x} + a_2 b_{2x} + a_3 b_{3x}) + \mathbf{j}\,(a_1 b_{1y} + a_2 b_{2y} + a_3 b_{3y}) + \mathbf{k}\,(a_1 b_{1z} + a_2 b_{2z} + a_3 b_{3z})\right] \\
&\quad + \left[\mathbf{i}\,(b_1 a_{1x} + b_2 a_{2x} + b_3 a_{3x}) + \mathbf{j}\,(b_1 a_{1y} + b_2 a_{2y} + b_3 a_{3y}) + \mathbf{k}\,(b_1 a_{1z} + b_2 a_{2z} + b_3 a_{3z})\right] \qquad \text{(as above)}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\mathbf{B}}(\mathbf{A} \cdot \mathbf{B}) &= \nabla_{\mathbf{B}}\,(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= \mathbf{i}\left(a_1\frac{\partial}{\partial x}b_1 + a_2\frac{\partial}{\partial x}b_2 + a_3\frac{\partial}{\partial x}b_3\right) + \mathbf{j}\left(a_1\frac{\partial}{\partial y}b_1 + a_2\frac{\partial}{\partial y}b_2 + a_3\frac{\partial}{\partial y}b_3\right) + \mathbf{k}\left(a_1\frac{\partial}{\partial z}b_1 + a_2\frac{\partial}{\partial z}b_2 + a_3\frac{\partial}{\partial z}b_3\right) \\
&= \left[\mathbf{i}\,(a_1 b_{1x} + a_2 b_{2x} + a_3 b_{3x}) + \mathbf{j}\,(a_1 b_{1y} + a_2 b_{2y} + a_3 b_{3y}) + \mathbf{k}\,(a_1 b_{1z} + a_2 b_{2z} + a_3 b_{3z})\right]
\end{aligned}$$

which equals the first bracket of the LHS. Similarly, it can be shown $\nabla_{\mathbf{A}}(\mathbf{A} \cdot \mathbf{B})$ equals the second. QED

Vector cross product:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$