Parametric Curve

Slope: $C : x = x(t), \quad y = y(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\,, \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \text{where } t = t(x), \text{as an inverse function of } x = x(t).$$

Concavity: $C : x = x(t), \quad y = y(t)$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{\frac{d^2y}{dt^2}}{\frac{dx}{dt}}$$

Arc Length: $C : x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta.$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$$

or informally, let $ds = \sqrt{(dx)^2 + (dy)^2}$,

so $L = \int_{t=\alpha}^{t=\beta} ds = \int_\alpha^\beta \frac{ds}{dt}\, dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$

Non-parametric form: $L = \int_{x=x(\alpha)}^{x=x(\beta)} ds = \int_a^b \frac{ds}{dx}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ where $a = x(\alpha)$ and $b = x(\beta)$.

Area of Surfaces of Revolution:

$C : x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta.$

$A =$ Area of C revolving around the x-axis between a and $b = \int_{t=\alpha}^{t=\beta} 2\pi\, y(t)\, ds$

$$= 2\pi \int_\alpha^\beta y(t)\, \frac{ds}{dt}\, dt = 2\pi \int_\alpha^\beta y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$$

If C is revolving around the y-axis, $A = 2\pi \int_\alpha^\beta x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$

Non-parametric form: $A = 2\pi \int_{x=x(\alpha)}^{x=x(\beta)} y(t)\, ds = 2\pi \int_a^b y(t)\, \frac{ds}{dx}\, dx = 2\pi \int_a^b y(t) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\,,$

where $a = x(\alpha)$ and $b = x(\beta)$.

Parametrisation in terms of arc length:

$C : x = f(\tau), \quad y = g(\tau), \quad \alpha \le \tau \le \beta\,.$

Let the arc length function be $\phi(t) = \int_\alpha^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2}\, d\tau, \quad$ where $\alpha \le t \le \beta$.

$\phi'(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2}, \quad \phi(\alpha) = 0$ and $\phi(\beta) = L\,.$

Let $s = \phi(t), \quad$ so $t = \phi^{-1}(s)$.

$C_\phi : x = f(\phi^{-1}(s)), \quad y = g(\phi^{-1}(s)), \quad 0 \le s \le L.$