# Ordinary Differential Equations (ODE)

**General Form:** $\displaystyle\sum_{k=0}^{n} f_k(x,y)\frac{d^k y}{dx^k} = G(x,y).\quad \left(\frac{d^0 y}{dx^0} \equiv y\right).$

For convenience, it is usually reduced to $f_n(x,y) = 1$. This is implied in this text.

**First-Order ODEs:** $\dfrac{dy}{dx} = F(x,y), \quad \big(\text{i.e. } F(x,y) = G(x,y) - f_0(x,y)y\big).$

Separable: $\dfrac{dy}{dx} = f(x)g(y).$ Solution: $\displaystyle\int \frac{dy}{g(y)} = \int f(x)dx + c.$

Linear: $\dfrac{dy}{dx} + P(x)y = Q(x).$ $\big($i.e. $f_0(x,y) = P(x)$, $G(x,y) = Q(x)$, both independent of $y$.$\big)$

Integrating factor $I = e^{\int P(x)dx}$. Solution: $\displaystyle\frac{d}{dx}(Iy) = I\frac{dy}{dx} + IP(x)y = IQ(x), \quad y = \frac{1}{I}\left(\int IQ(x)dx + c\right).$

**Second-Order ODEs:** $\dfrac{d^2y}{dx^2} + f_1(x,y)\dfrac{dy}{dx} + f_0(x,y)y = G(x,y).$

Homogeneous $\big(G(x,y) = 0\big)$ with Constant Coefficients: $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 0.$

Solutions form a 2-dimensional vector space: $y = \alpha_1 y_1(x) + \alpha_2 y_2(x)$,

where $y_1$ and $y2$ are linearly independent. If they are linearly dependent, $y = (\alpha_1 + \alpha_2 x)y_1(x)$.

Function $y = e^{\lambda x}$ will make $\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0.$ $\lambda^2 + a\lambda + b = 0.$

For two distinct real roots: $y = \alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x}$. For two equal real roots $\left(\lambda = \dfrac{-a}{2}\right)$: $y = (\alpha_1 + \alpha_2 x)e^{\frac{-ax}{2}}$.

For two complex roots $\left(\dfrac{-a}{2} \pm iw, \text{ where } w = \sqrt{b - \dfrac{a^2}{4}}\right)$: $y = \alpha_1 e^{(\frac{-a}{2}+iw)x} + \alpha_2 e^{(\frac{-a^2}{2}-iw)x}$.

$y = e^{\frac{-ax}{2}}\left(\alpha_1 e^{iwx} + \alpha_2 e^{-iwx}\right) = e^{\frac{-ax}{2}}\left(\alpha_1(\cos(wx) + i\sin(wx)) + \alpha_2(\cos(wx) - i\sin(wx))\right).$

$y = e^{\frac{-ax}{2}}\left(\beta_1\cos(wx) + \beta_2\sin(wx)\right),$ where $\beta_1$ and $\beta_2$ are two arbitrary complex numbers.

Free Oscillations: Force = - friction - elasticity, $ma = -cv - ky$, or $m\dfrac{d^2y}{dt^2} + c\dfrac{dy}{dt} + ky = 0, \quad (m, c, k > 0).$

Overdamping (large friction $c^2 > 4mk$): $y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$. $\displaystyle\lim_{t\to\infty} y = 0.$

Critical damping ($c^2 = 4mk$): $y = (A + Bt)e^{-\frac{ct}{2m}}$. $\displaystyle\lim_{t\to\infty} y = 0.$

Underdamping ($c^2 < 4mk$): $y = \left(A\cos(\Omega t) + B\sin(\Omega t)\right)e^{-\alpha t}$, where $\alpha = \dfrac{c}{2m}$ and $\Omega = \sqrt{\dfrac{k}{m} - \dfrac{c^2}{4m^2}}$.

For a right angle triangle with sides $A$, $B$, and angle $\delta$ between $R$ and $A$, $R = \sqrt{A^2 + B^2}$ and $\tan(\delta) = \dfrac{B}{A}$.

$y = R\left(\dfrac{A}{R}\cos(\Omega t) + \dfrac{B}{R}\sin(\Omega t)\right)e^{-\alpha t} = R\left(\cos(\delta)\cos(\Omega t) + \sin(\delta)\sin(\Omega t)\right)e^{-\alpha t}.$

$y = Re^{-\alpha t}\cos(\Omega t - \delta)$, a decaying oscillation if $c > 0$.

If $c = 0$ (no friction), $y = R\cos(\Omega_0 t - \delta)$, where $\Omega_0 = \sqrt{\dfrac{k}{m}}$. The period is $\dfrac{2\pi}{\Omega_0}$.