The Laplace Transform

Definition: $\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)dt$, where $f(t)$ is a given function defined for all $t \geq 0$. $f(t) = \mathcal{L}^{-1}(F)$.

It can be proven that $\mathcal{L}^{-1}(F)$ is unique if null function is excluded.

Basics:

Linearity: $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$.

Derivatives: $\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}(f) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$. $(f^{(0)} \equiv f.)$

If $f^{(k)} = 0$ for $k = 0, 1 \ldots, n-1$, $\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}(f)$, turning a differential expression into a polynomial.

Integral: $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}(f(t)) = \frac{F(s)}{s}$. $\mathcal{L}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)d\tau$.

Proof: Let $g(t) = \int_0^t f(\tau)d\tau$, $\mathcal{L}(f(t)) = \mathcal{L}(g'(t)) = s\mathcal{L}(g) - g(0) = s\mathcal{L}(g)$, $\mathcal{L}(g) = \frac{1}{s}\mathcal{L}(f)$.

Shift on s-axis: $\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)$.

Shift on t-axis: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$,

where $a \geq 0$ and the step function $u(t) = 0$ if $t < 0$, $u(t) = \frac{1}{2}$ if $t = 0$, $u(t) = 1$ if $t > 0$.

Partial Fractions: $Y(s) = \frac{F(s)}{G(s)}$, where $F(s)$ and $G(s)$ are polynomials in s, and deg(F)<def(G).

Read factors: If $G(s) = (s-a)^m (s-b)^n \ldots$

then $Y(s) = \frac{F(s)}{G(s)} = \sum_{k=1}^{m} \frac{A_k}{(s-a)^k} + \sum_{k=1}^{n} \frac{B_k}{(s-b)^k} + \ldots$ where A_k and B_k are constants.

For $Y_a(s) = \sum_{k=1}^{m} \frac{A_k}{(s-a)^k}$, $(s-a)^m\, Y_a(s) = \sum_{k=1}^{m} A_k (s-a)^{m-k}$.

$$\frac{d^{m-k}}{ds^{m-k}}\big((s-a)^m\, Y_a(s)\big) = \sum_{r=1}^{m} A_r \frac{d^{m-k}}{ds^{m-k}}(s-a)^{m-r} = \sum_{r=1}^{k} A_r \frac{(m-r)!}{(k-r)!}(s-a)^{k-r} \quad \text{for } k > 1, \text{ or } A_1(m-1)! \text{ when } k = 1.$$

$\lim_{s\to a} \text{RHS} = A_k(m-k)!$, $A_k = \frac{1}{(m-k)!}\lim_{s\to a}\left(\frac{d^{m-k}}{ds^{m-k}}\big((s-a)^m\, Y_a(s)\big)\right)$. For $k = m$, it is simply $A_m = \lim_{s\to a}(s-a)^m Y_a(s)$.

For $Y_b(s) = \sum_{k=1}^{n} \frac{B_k}{(s-b)^k}$, $\lim_{s\to a}\left(\frac{d^{m-k}}{ds^{m-k}}\big((s-a)^m\, Y_b(s)\big)\right) = 0$ for $1 <= k <= m$.

$$\boxed{A_k = \frac{1}{(m-k)!}\lim_{s\to a}\left(\frac{d^{m-k}}{ds^{m-k}}\big((s-a)^m\, Y(s)\big)\right) \quad \text{for } k = 1, \ldots, m-1 \quad \text{and } A_m = \lim_{s\to a}(s-a)^m Y(s).}$$

$$\mathcal{L}^{-1}\left(\frac{A_k}{(s-a)^k}\right) = A_k \mathcal{L}^{-1}\left(\frac{1}{(s-a)^k}\right) = A_k\left(e^{at}\frac{t^{k-1}}{(k-1)!}\right).$$

$$\boxed{\mathcal{L}^{-1}\big(Y(s)\big) = e^{at}\left(\sum_{k=1}^{m} \frac{A_k}{(k-1)!} t^{k-1}\right) + \ldots}$$

When $m = 1$ (non-repeating factors), $A_1 = \lim_{s\to a} \frac{(s-a)}{G(s)} F(s) = \lim_{s\to a} \frac{(s-a)}{G(s) - G(a)} F(s) = \frac{F(a)}{G'(a)}$.

$$\boxed{\mathcal{L}^{-1}\big(Y(s)\big) = \frac{F(a)}{G'(a)} e^{at} + \ldots}$$

Initial Value Theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty}(sF(s))$.

Final Value Theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0}(sF(s))$.

Two complex factors:

If $\mathbf{G(s)} = \mathbf{(s-a)(s-\bar{a})}$ and $a = \alpha + i\beta$, then $G(s) = s^2 - 2s\alpha + \alpha^2 + \beta^2 = (s-\alpha)^2 + \beta^2$.

$$Y(s) = \frac{F(s)}{G(s)} = \frac{As+B}{(s-\alpha)^2+\beta^2} = \frac{A(s-\alpha)+\alpha A+B}{(s-\alpha)^2+\beta^2}.$$

$$\boxed{\mathcal{L}^{-1}\big(Y(s)\big) = e^{\alpha t}\big(\mathrm{Im}(Q_a)\cos\beta t + \mathrm{Re}(Q_a)\sin\beta t\big), \quad \text{where } Q_a = \frac{1}{\beta}\lim_{s\to a}\frac{[(s-\alpha)^2+\beta^2]F(s)}{G(s)}.}$$

If $\mathbf{G(s)} = \mathbf{[(s-a)(s-\bar{a})]^2}$ and $a = \alpha + i\beta$, then $G(s) = [(s-\alpha)^2+\beta^2]^2$.

$$Y(s) = \frac{F(s)}{G(s)} = \frac{As+B}{[(s-\alpha)^2+\beta^2]^2} + \frac{Cs+D}{(s-\alpha)^2+\beta^2}\ .$$

$$\boxed{\mathcal{L}^{-1}\big(Y(s)\big) = e^{\alpha t}\left[\frac{A}{2\beta}t\sin\beta t + \frac{\alpha A+B}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)\right] + e^{\alpha t}\left[C\cos\beta t + \frac{\alpha C+D}{\beta}\sin\beta t\right].}$$

Ordinary Differential Equations:

Example: $ay'' + by = h(t), \quad \mathcal{L}(ay''+by) = \mathcal{L}(h(t)), \quad a[s^2\mathcal{L}(y) - sy(0) - y'(0)] + b\mathcal{L}(y) = \mathcal{L}(h(t)),$

$$(as^2+b)\mathcal{L}(y) = \mathcal{L}(h(t)) + asy(0) + ay'(0), \quad y = \mathcal{L}^{-1}\left[\frac{\mathcal{L}(h(t)) + asy(0) + ay'(0)}{as^2+b}\right].$$