Hyperbolic Functions

Revision: Consider circle $x^2 + y^2 = 1$. The line segment from the origin to a point on the circle (x, y) form an angle α with the x-axis. The area bound by this line segment, the x-axis and the circle is $\frac{\alpha}{2}$.
We define: $\cos\alpha = x$ and $\sin\alpha = y$.

Definition: Consider hyperbola $x^2 - y^2 = 1$. Since there is no "angle at the centre" for a hyperbola, we define α such that $\frac{\alpha}{2}$ equals to the area bound by the x-axis, the hyperbola and the line segment joining the origin and a point (x, y) on the hyperbola. We define: $\cosh\alpha = x$ and $\sinh\alpha = y$.

Let us consider the first quadrant only, where the hyperbola can be expressed by $y = \sqrt{x^2 - 1}$.

The area $\dfrac{\alpha}{2} = \dfrac{1}{2}xy - \displaystyle\int_1^x y\ dx = \dfrac{1}{2}xy - \int_1^x \sqrt{x^2-1}\ dx$

$$= \frac{1}{2}xy - \left[\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right|\right]_1^x$$

$$= \frac{1}{2}x\sqrt{x^2-1} - \left(\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right|\right)$$

$$= \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right|$$

$\therefore \quad \alpha = \ln\left(x + \sqrt{x^2-1}\right)$ for $x, y \geq 1$. $\qquad e^\alpha = x + \sqrt{x^2-1}$.

$(e^\alpha - x)^2 = x^2 - 1, \quad e^{2\alpha} - 2e^\alpha x + x^2 = x^2 - 1, \quad \cosh\alpha = x = \dfrac{e^{2\alpha}+1}{2e^\alpha} = \dfrac{e^\alpha + e^{-\alpha}}{2}$.

$$\sinh\alpha = y = \sqrt{x^2-1} = \sqrt{\frac{(e^\alpha + e^{-\alpha})^2}{4} - 1} = \sqrt{\frac{e^{2\alpha} + e^{-2\alpha} + 2 - 4}{2}} = \sqrt{\frac{(e^\alpha - e^{-\alpha})^2}{4}} = \frac{e^\alpha - e^{-\alpha}}{2}.$$

$\therefore \quad \boxed{\cosh\alpha = \dfrac{e^\alpha + e^{-\alpha}}{2}, \quad \sinh\alpha = \dfrac{e^\alpha - e^{-\alpha}}{2}}$ which can be easily extended to all four quadrants.

Likewise, $\tanh\alpha = \dfrac{\sinh\alpha}{\cosh\alpha} = \dfrac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}}, \quad \coth\alpha = \dfrac{\cosh\alpha}{\sinh\alpha} = \dfrac{e^\alpha + e^{-\alpha}}{e^\alpha - e^{-\alpha}}$, etc.

Imagine... It follows that $\cosh^2 x - \sinh^2 x = 1$, which links to the "circular" functions: $\cos^2 x + \sin^2 x = 1$.
Since $i^2 = -1$, you may imagine that the hyperbolic and "circular" functions are related through i.

Consider $\cosh(i\alpha) = \dfrac{e^{i\alpha} + e^{-i\alpha}}{2} = \dfrac{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)}{2} = \cos\alpha$,

and $\sinh(i\alpha) = \dfrac{e^{i\alpha} - e^{-i\alpha}}{2} = \dfrac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{2} = i\sin\alpha$,

We define: $\boxed{\cos(i\alpha) = \cosh\alpha \quad \text{and} \quad \sin(i\alpha) = i\sinh\alpha}$ so $\cosh\alpha = \cos(i\alpha)$ and $\sinh\alpha = -i\sin(i\alpha)$.

It follows: $\tan(i\alpha) = \dfrac{\sin(i\alpha)}{\cos(i\alpha)} = \dfrac{i\sinh\alpha}{\cosh\alpha} = i\tanh\alpha$ and $\cot(i\alpha) = \dfrac{\cos(i\alpha)}{\sin(i\alpha)} = \dfrac{\cosh\alpha}{i\sinh\alpha} = -i\coth\alpha$

Now the equivalence is clear: $1 = \cos^2(i\alpha) + \sin^2(i\alpha) = \cosh^2\alpha + (i\sinh\alpha)^2 = \cosh^2\alpha - \sinh^2\alpha$.

You may imagine a rectangular hyperbola with its foci on x-axis being a circle with an imaginary y-axis (conceptually at $x = \pm\infty$). The angle at its (far away) "centre" can be represented by $2i$ times the area bound by the x-axis, the hyperbola and line segment $(0, 0) - (x, y)$. The y coordinate of this imaginary circle (given by $\cos(i\alpha)$) is iy (or $i\sinh\alpha$ in the context of the hyperbola). On this notion, hyperbolic functions are equivalent to their "circular" counterparts on imaginary y and angle measurements.

To take it further

Formulae of hyperbolic functions can be easily derived from their "circular" counterparts.
Given $\cosh x = \cos(ix)$, $\sinh x = -i\sin(ix)$, $\tanh x = -i\tan(ix)$,
$\coth x = i\cot(ix)$, $\text{sech } x = \sec(ix)$, and $\text{csch } x = i\csc(ix)$, we have...

$$\sinh(-x) = -i\sin(-ix) = i\sin(ix) = -\sinh x,$$
$$\cosh(-x) = \cos(-ix) = \cos(ix) = \cosh x,$$
$$\tanh(-x) = -i\tan(-ix) = i\tan(ix) = -\tanh x,$$

$$\cosh^2 x - \sinh^2 x = \cos^2(ix) - (-i\sin(ix))^2 = \cos^2(ix) + \sin^2(ix) = 1,$$
$$1 - \text{sech}^2 x = 1 - \sec^2(ix) = -\tan^2(ix) = -(i\tanh x)^2 = \tanh^2 x,$$
$$1 + \text{csch}^2 x = 1 + (i\csc(ix))^2 = 1 - \csc^2(ix) = -\cot^2(ix) = -(-i\coth x)^2 = \coth^2 x,$$

$$\sinh(x+y) = -i\sin(ix+iy) = -i[\sin(ix)\cos(iy) + \cos(ix)\sin(iy)] = \sinh x\cosh y + \cosh x\sinh y,$$
$$\cosh(x+y) = \cos(ix+iy) = \cos(ix)\cos(iy) - \sin(ix)\sin(iy) = \cosh x\cosh y + \sinh x\sinh y,$$
$$\tanh(x+y) = -i\tan(ix+iy) = -i\frac{\tan(ix)+\tan(iy)}{1-\tan(ix)\tan(iy)} = \frac{-i\tan(ix) - i\tan(iy)}{1+(-i)\tan(ix)(-i)\tan(iy)} = \frac{\tanh x + \tanh y}{1+\tanh x\tanh y},$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\cos(ix) = -i\sin(ix) = \sinh x,$$
$$\frac{d}{dx}\sinh x = \frac{d}{dx}(-i\sin(ix)) = -i(i\cos(ix)) = \cos(ix) = \cosh x,$$
$$\frac{d}{dx}\tanh x = \frac{d}{dx}(-i\tan(ix)) = -i(i\sec^2(ix)) = \text{sech}^2 x.$$

For inverse hyperbolic functions, let us try $\tanh^{-1} x$:

Let $u = \tanh^{-1} x$. $x = \tanh u = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$, $x(e^u + e^{-u}) = e^u - e^{-u}$, $e^u(1-x) = e^{-u}(1+x)$,

$u + \ln(1-x) = -u + \ln(1+x)$, $2u = \ln(1+x) - \ln(1-x)$, $\tanh^{-1} x = u = \dfrac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right|$.

Let $v = \tan^{-1}(ix)$. $ix = \tan v = \tan(i(-iv)) = i\tanh(-iv)$, $x = \tanh(-iv)$,

so $\tanh^{-1} x = -iv = -i\tan^{-1}(ix)$.

This can be used in integration. For example, given $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$, find $\displaystyle\int \frac{1}{1-x^2}\,dx$:

$$\int \frac{1}{1-x^2}\,dx = \int \frac{1}{1+(ix)^2}\,dx = \frac{1}{i}\tan^{-1}(ix) + C = -i\tan^{-1}(ix) + C$$
$$= \tanh^{-1} x + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C.$$

No wonder in integration formulae, negating the sign of x^2 often causes the inverse trigonometric function in the result to change to a logarithm function.