Fourier Series

Periodic Functions: $f(x+p) = f(x)$ for all x.

If f is not constant, the smallest positive p for which the above holds is called the **primitive period** of f.

Fourier Series: $\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$, a periodic function with primitive period 2π.

To find the coefficients a_n and b_n:

$$\int_{-\pi}^{\pi} f(x)\ dx = \int_{-\pi}^{\pi} a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\ dx = \int_{-\pi}^{\pi} a_0\ dx = 2\pi a_0.$$

$$\therefore \quad \boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx.}$$

$$\because \quad \int_{-\pi}^{\pi} f(x) \cos mx\ dx = \pi a_m, \quad \therefore \text{ we have } \quad \boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\ dx, \quad m \in \mathbb{N}.}$$

Proof: $$I = \int_{-\pi}^{\pi} f(x)\cos mx\ dx = \int_{-\pi}^{\pi} a_0 \cos mx\ dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx\ dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx\ dx\right]$$

$$= \int_{-\pi}^{\pi} a_0 \cos mx\ dx + \sum_{n=1}^{\infty}\left[a_n\ \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x\ dx + a_n\ \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x\ dx\right.$$

$$\left. + b_n\ \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x\ dx + b_n\ \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x\ dx\right].$$

All terms are zero except $\frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\ dx = \pi$ when $n = m$. $\quad \therefore I = \pi a_m.$

Likewise, $\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\ dx, \quad m \in \mathbb{N}.}$

For periodic function with period $2L$, $\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).}$

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx.}$$

$$\boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\ dx, \quad m \in \mathbb{N}.}$$

$$\boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\ dx, \quad m \in \mathbb{N}.}$$

Complex Form: $\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n\ e^{inx}}$, where $\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}\ dx.}$

Note: The resulting f is still a real function.

Linearlity: If f_1 and f_2 are two Fourier Series, $af_1 + bf_2$ is a Fourier Series with coefficients equal to the sums of the corresponding coefficients of $f1$ and $f2$ times a and b respectively.

i.e. Let $f_1 = \sum_{n=-\infty}^{\infty} c_{1n}\ e^{inx}$ and $f_2 = \sum_{n=-\infty}^{\infty} c_{2n}\ e^{inx}$, $af_1 + bf_2 = \sum_{n=-\infty}^{\infty} (a\ c_{1n} + b\ c_{2n})\ e^{inx}$.

If f is a Fourier Series, kf is a Fourier Series with coefficients

equal to k times the corresponding coefficients of f. i.e. $kf = \sum_{n=-\infty}^{\infty} (k\ c_n\ e^{inx})$.